REALIZING INTERVAL GRAPHS WITH SIZE AND DISTANCE CONSTRAINTS

ITSIK PE'ER AND RON SHAMIR

Abstract. We study the following problem: Given an interval graph, does it have a realization which satisfies additional constraints on the distances between interval endpoints? This problem arises in numerous applications in which topological information on intersection of pairs of intervals is accompanied by additional metric information on their order, distance or sizes. An important application is physical mapping, a central challenge in the human genome project. Our results are: (1) A polynomial algorithm for the problem on interval graphs which admit a unique clique order (UCO graphs). This class of graphs properly contains all prime interval graphs. (2) In case all constraints are upper and lower bounds on individual interval lengths, the problem on UCO graphs is linearly equivalent to deciding if a system of difference inequalities is feasible. (3) Even if all the constraints are prescribed lengths of individual intervals, the problem is NP-complete. Hence, problems (1) and (2) are also NP-complete on arbitrary interval graphs.

1. Introduction. A graph $G(V,E)$ is an *interval graph* if one can assign to each vertex v an interval I_v on the real line, so that two intervals have a non-empty intersection if and only if their vertices are adjacent. The set of intervals $\{I_v\}_{v\in V}$ is called a *realization* of G. The problems which we study here are concerned with the existence of an interval realization to a graph, subject to various types of *distance (or difference) constraints* on interval endpoints. These are inequalities of the form $x-y<C_{xy}$ or $x-y\le C_{xy}$, for variables x,y and constant C_{xy}. Specifically, we study the following problems (we defer further definitions to section 2):

> Distance-Constrained Interval Graph ($DCIG$):
> **INSTANCE:** A graph $G=(V,E)$ and a system S of distance constraints on the variables $\{l_v,r_v\}_{v\in V}$.
> **QUESTION:** Does G have a closed interval realization whose endpoints satisfy S? That is, is there a set of intervals $\{[l_v,r_v]|v\in V\}$ which form a realization of G and their endpoints satisfy S?

A special case is $DCIG$ in which all constraints are lower and upper bounds on interval lengths:

> Bounded Interval Graph Recognition (BIG):
> **INSTANCE:** A graph $G=(V,E)$, and functions $L:V\to\mathbb{N}$, $U:V\to\mathbb{N}$.
> **QUESTION:** Is there a closed interval realization of G such that for each vertex v: $L(v)\le|I_v|\le U(v)$.

In the following problem, each interval must have a prescribed length:

> Measured Interval Graph Recognition (MIG^*):
> **INSTANCE:** A graph $G=(V,E)$ and a *length* function $L:V\to\mathbb{N}$.
> **QUESTION:** Is there a closed interval realization of G, in which for every $v\in V$,

$|I_v| = L(v)$?

We shall prove here that even MIG^*, the most restricted problem of the three, is strongly NP-complete. Unlike the situation with interval graphs, the fact that the intervals must be closed causes some loss in generality. In contrast, we show that when the interval graph admits a unique consecutive clique order (up to complete reversal), $DCIG$ is polynomial, and hence, so are the other two problems. The class of graphs satisfying this property (which we call UCO graphs) properly contains the class of prime interval graphs, and is recognizable in linear time. Our solution is based on reducing the problem to a system of difference constraints. We also prove that we cannot do better, by showing that the problem of solving a system of difference constraints and the problem BIG on UCO graphs are linearly equivalent.

Interval graphs have been intensively studied, due to their central role in many applications (cf. [33, 17, 11]). They arise in many practical problems which require the construction of a time line where each particular event or phenomenon corresponds to an interval representing its duration. Among the applications are planning [3], scheduling [22, 31], archaeology [26], temporal reasoning [2], medical diagnosis [29], and circuit design [36]. There are also non-temporal applications in genetics [6] and behavioral psychology [9]. In the Human Genome Project, a central problem which bears directly on interval graphs is the physical mapping of DNA [8, 25]: It calls for the reconstruction of a map (a realization) for a collection of DNA segments, based on information on the pairwise intersections of segments.

In the applications above, size and distance constraints on the intervals may occur naturally: The lengths of events (intervals) may be known precisely, or may have upper and lower bounds. The order or distance between two events may be known. This is often the case in scheduling problems and temporal reasoning. In physical mapping, certain experiments provide information on the sizes of the DNA segments [21]. Our goal here is to study how to combine those additional constraints with precise intersection data.

Green and Xu (cf. [20]) developed and implemented a program (called SEGMAP) for construction of physical maps of DNA, which utilizes intersection and size data. The intersection data is obtained by experimentally testing whether each of the segments contain a sequence of DNA (called STS) which appears in a unique, unknown location along the chromosome. Hence, two segments which contain a common STS must intersect. Their algorithm works in two phases: the first phase ignores the size data. It obtains a partition of the STSs into groups, and a linear order on the groups. The second phase uses the partial order of phase 1 together with the size data to obtain the map using linear programming algorithms. Our results in section 3 imply that faster algorithms (utilizing network flow techniques) can be used under certain conditions on the data. However, the results in section 5 imply that the general problem tackled by SEGMAP is intractable (unless P=NP) even with perfect data.

Recognizing interval graphs (i.e., deciding if a graph has an interval realization) can be done in linear time [7, 28, 23]. Surprisingly, much less is known about the realization problem when the input contains additional constraints on the realization. The special case of MIG where all intervals have equal (unit) length corresponds to recognizing *unit interval graphs* [33], which can be done in linear time [10]. The special case of $DCIG$ where all distance constraints have the form $r_v - l_u < 0$ or $l_v - r_u \leq 0$ is the problem of *seriation with side constraints* [27, 19] (also called *interval graph with order constraints*), which can also be solved in linear time [32]. When $DCIG$ is further restricted to the special case where for each pair u, v where $(u, v) \notin E$, we have either the constraint $r_v - l_u < 0$ or $r_u - l_v < 0$. The problem is equivalent to recognizing an interval order, which can be done in linear time [4]. Fishburn and Graham [12] discussed a special case of BIG where all intervals have the same pair p and q of upper and lower bounds. For each p and q, they characterized the resulting class of interval graphs (and interval orders), in terms of the family of minimal forbidden induced subgraphs (respectively, suborders). They proved that such a family is finite if and only if $\frac{p}{q}$ is rational. In this case, for integer p and q, their characterization yields an exponential time $n^{O(pq)}$ algorithm for identification of such graphs (orders), where n is the number of vertices. Isaak [24] studied a variant of BIG in which the input is an interval order, there are upper and lower integer bounds on individual interval lengths, and the question is whether there exist a realization in which all endpoints are integers. Using Bellman's notion of a distance graph, Isaak gave an $O(\min(n^3, n^{2\frac{1}{2}} \log nC))$ time algorithm for that problem, where C is the sum of bounds on lengths. He also posed the more general problem of BIG, which we answer here. We generalize distance graphs to handle both strict and weak inequalities on endpoints, in order to solve $DCIG$ on a particular class of graphs.

There have been other studies on the realization of a set of intervals based on partial information on their intersection, length and order. Those are different from our problems here inasmuch the information on intersection is incomplete, i.e., the underlying interval graph is not completely known. Among these are studies on interval sandwich [18], interval satisfiability [19, 37, 32], on interval graphs and orders which have realizations with at most k different lengths [11, chapter 9], on the smallest interval orders whose representation requires at least k different lengths [11, chapter 10], and on the number of distinct interval graphs and orders on n vertices which have a realization with k given lengths [35].

The paper is organized as follows: Section 2 contains some preliminaries and background. Section 3 studies problem $DCIG$ on UCO graphs, and proves its linear equivalence to solving systems of difference constraints. This implies in particular an $O(\min(n^3, n^{2\frac{1}{2}} \log nC))$ time algorithm for all three problems on UCO graphs. In section 4 we sketch a simple proof that $DCIG$ is strongly NP-complete. Section 5 proves the stronger result that MIG^* is strongly NP-complete. The reduction (performed in two steps) is rather involved, but we feel it gives insight on the interplay between the topological side of the problem (i.e., intersection, open or closed intervals) and its metric aspect (i.e., the intervals sizes).

2. Preliminaries. A graph $G = (V, E)$ is called an *intersection graph* of a family of sets $S = \{I_v\}_{v \in V}$, if $I_v \cap I_u \neq \emptyset \Leftrightarrow vu \in E$. G is called an *interval graph* if it is an intersection graph of a family $S = \{I_v\}_{v \in V}$ of intervals on the real line. In that case, S is called a *realization* of G. Depending on the convention, each interval may be either closed or open, with no loss of generality. For simplicity, we sometimes use the same names for the intervals and for the corresponding vertices.

For an interval I denote its left and right endpoints by $l(I)$ and $r(I)$, respectively. The *length* of I, denoted $|I|$, is $r(I) - l(I)$. If G has a realization in which all the intervals are of equal length, then it is called a *unit interval graph*.

Let $C_1, \ldots, C_k$ be the maximal cliques in a graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$. The ***clique matrix*** of G is the $n \times k$ zero-one matrix $C(G) = (m_{ij})$ where $m_{ij} = 1$ if and only if $v_i \in C_j$. If the columns in $C(G)$ can be permuted so that the ones in each row are consecutive, then we say that $C(G)$ has the *consecutive ones* property, and we call such a permutation of the columns a *consecutive (clique) order*. According to Gilmore and Hoffman [16], G is an interval graph if and only if $C(G)$ has the consecutive ones property.

For two non-intersecting intervals x, y where x is completely to the left of y, we write $x \prec y$ or, equivalently, $y \succ x$. Let $P = (V, <)$ be a partial order. Call $<$ an *interval order* if there exists a set of intervals $S = \{I_v\}_{v \in V}$ such that $v < u$ if and only if $I_v \prec I_u$. S is called a *realization* for P. Call $G = (V, E)$ the *incomparability graph* of P, if for each $u, v \in V$, $uv \in E$ if and only if u and v are incomparable in P, i.e., $u \not< v$ and $v \not< u$. Hence, G is an interval graph if and only if it is the incomparability graph of some interval order. In this case we will say that the graph G ***admits*** the order $<$.

For a vertex $v \in V$ in the graph $G = (V, E)$, denote $N(v) = \{u \in V | uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. For a vertex set $U \subseteq V$ denote $N[U] = \cup_{u \in U} N[u]$ and $N(U) = N[U] \setminus U$. A set $M \subseteq V$ is called a *module* in $G = (V, E)$ if for each $x, y \in M$, and for each $u \notin M$: $xu \in E \Leftrightarrow yu \in E$. Surely, V is a module, and for each $v \in V$, $\{v\}$ is a module. Such modules are called *trivial*. If all modules in G are trivial, then G is called *prime*. For a subset $X \subset V$ define $E[X] = \{uv \in E | u, v \in X\}$. For a module M in the graph G, the graph $G' = (V', E')$, where $V' = (V \setminus M) \cup \{v\}$, and $E' = E[V \setminus M] \cup \{uv | u \in N(M)\}$, is said to be *obtained from G by contracting M to v*. We usually denote by n and m the number of vertices and edges, respectively, in the graph.

3. Distance Constraints in UCO graphs. We call an interval graph *uniquely clique-orderable* (UCO for short) if it has a unique consecutive clique order, up to complete reversal, in every realization. An interval graph G is UCO if and only if the only non-trivial modules in it are cliques [34]. Note that G is UCO if and only if the interval order admitted by G is unique, up to complete reversal, because an interval order of the vertices of G uniquely determines a linear order of the maximal cliques in G, and vice versa. Denote this order by $\prec_G$. Note also that the class of UCO graphs properly contains the class of prime interval graphs. UCO graphs can be recognized in linear time by applying the PQ-tree algorithm of Booth and Lueker [7], and noting that G is UCO if and only if the final tree consists of a single internal Q-node and the leaves. This procedure also computes $\prec_G$ in $O(m+n)$ time.

In this section we study the problem $DCIG$ when the input graph is UCO. We show how to reduce this problem, in linear time, to the problem of deciding whether a system of difference constraints is feasible. Hence, $DCIG$, BIG and MIG are all polynomial on UCO graphs. We also prove that for BIG and $DCIG$, we cannot do any better, since deciding the feasibility of a system of difference constraints can be reduced in linear time to an instance of BIG with a UCO graph.

3.1. A Polynomial Algorithm for $DCIG$ on UCO Graphs. Let $P = (G, A)$ be an instance of $DCIG$, where $G = (V, E)$ is UCO and A is a set of difference inequalities on the interval endpoints.

Construct two systems T and $\bar{T}$ of difference constraints on the variables $\{l_v, r_v\}_{v\in V}$, as follows: Both systems include all inequalities in A. In addition, for each $x, y \in V$: If $x \prec_G y$ then T contains an inequality $r_x < l_y$, and $\bar{T}$ contains an inequality $r_y < l_x$. If $xy \in E$ then both T and $\bar{T}$ contain an inequality $r_x \geq l_y$ (and $r_y \leq l_x$). With these definitions we prove:

LEMMA 3.1. *P has a realization if and only if either T or $\bar{T}$ has a feasible solution.*

Proof. If $X = \{\tilde{l}_v, \tilde{r}_v\}_{v\in V}$ is a feasible solution to T or to $\bar{T}$, then X is a solution to A, and $\{[\tilde{l}_v, \tilde{r}_v]\}_{v\in V}$ realizes G. On the other hand, let $\{[\tilde{l}_v, \tilde{r}_v]\}_{v\in V}$ be a realization of G, whose endpoints satisfy A. Then the order of the intervals $\{[\tilde{l}_v, \tilde{r}_v]\}_{v\in V}$ on the real line is either $\prec_G$, or its reversal. Therefore, $\{\tilde{l}_v, \tilde{r}_v\}_{v\in V}$ is a feasible solution to either T or $\bar{T}$. □

Hence, we can solve our problem by deciding whether system T or $\bar{T}$ is feasible. We shall prove now that a system S of weak and strict difference constraints on n variables is reducible in linear time to a system S' which consists of weak difference constraints, with numbers only $O(n)$ times larger. (Standard transformation techniques [14] would give numbers $O(2^L)$ times larger for binary input length L.)

Assume all constants in S to be integral, and fix $\epsilon \leq \frac{1}{n}$. Define S' to include every weak inequality $x - y \leq c$ in S, and a weak inequality $x - y \leq c - \epsilon$ for every strict inequality $x - y < c$ in S. Note that the number of variables and number of inequalities in the two systems is the same, and the constants in S' (after multiplying by an appropriate factor to restore integrality) are larger than the constants in S by a factor of $\Theta(n)$.

LEMMA 3.2.

S has a feasible solution if and only if S' has one.

Proof. The 'if' direction is trivial, since a feasible solution to S' also satisfies S.

To prove the 'only if', we generalize the notion of a distance graph (cf. [1, p. 103]), to handle strict and weak inequalities: For a system T of difference constraints, construct a directed weighted graph $D(T) = (V, A)$, with arc weights and arc labels, as follows: For every constraint $x - y \leq C_{xy}$ or $x - y < C_{xy}$ add an arc (y, x) to $D(T)$ with weight C_{xy} and label the arc $\leq$ or $<$, respectively. $D(T)$ is called the *distance graph* of the system T. The *weight* of a path (or a cycle) in this graph is the sum of the weights of its arcs. Bellman has shown that when all inequalities in T are weak, T is feasible if and only if $D(T)$ contains no negative cycle ([5], see also [1, p. 103]).

Suppose S' is not feasible. Then $D(S')$ must contain a negative-weight cycle c. Let $w(c)$ and $w'(c)$ be the total weight of c in $D(S)$ and $D(S')$, respectively. Distinguish two cases:

- All arcs in c have labels $\leq$. Then $w(c) = w'(c) < 0$. But:

$$w(c) = \sum_{(y,x)\in c} C_{xy} \geq \sum_{(y,x)\in c} (x - y) = 0 \tag{1}$$

  Hence, S is infeasible.
- c contains an arc marked $<$. Since the weight of each arc in c differs from the weight of the

corresponding arc in c' by no more than ϵ, we get:

$$w(c) \leq w'(c) + n\epsilon \leq w'(c) + 1 < 1$$

Since the weights in $D(S)$ are integral, it follows that $w(c) \leq 0$. Since the cycle c in $D(S)$ contains an arc marked $<$, the inequality (1) is strict, namely, $w(c) > 0$, so S is infeasible.

□

COROLLARY 3.3.

A system T is feasible if and only if the weight of every cycle in its distance graph $D(T)$ is either positive, or it is zero and the cycle consists of $\leq$ arcs only. ■

We now show that addition of identical strict inequalities to the equivalent systems S and S' above maintains the equivalence between them. (We will need this property in section 5.3): For constants $\{C_i\}_{i \in I_1 \cup I_2 \cup I_3}$, define the following systems S_1, S_2, S_2' and S_3 on the set of variables $X = \{x_i\}_{i=1}^{n}$:

(S_1) $\qquad x_{j_i} - x_{k_i} \leq C_i \qquad i \in I_1$

(S_2) $\qquad x_{j_i} - x_{k_i} < C_i \qquad i \in I_2$

(S_2') $\qquad x_{j_i} - x_{k_i} \leq C_i - \epsilon \qquad i \in I_2$

(S_3) $\qquad x_{j_i} - x_{k_i} < C_i \qquad i \in I_3$

LEMMA 3.4. *Let $S = S_1 \cup S_2 \cup S_3$ and $S' = S_1 \cup S_2' \cup S_3$, where all C_i-s are integers and $\epsilon < \frac{1}{n}$. S has a feasible solution if and only if S' has one.*

Proof. The proof is by induction on the size of I_3. For $I_3 = \phi$, this is lemma 3.2. Suppose both S and S' have feasible solutions, and consider adding a single strict inequality E: $x - y < C$ to both systems. This implies adding an arc $e = yx$ labeled $<$ with $w(e) = C$ to both distance graphs $D(S)$ and $D(S')$. By corollary 3.3, it suffices to prove that there exist a cycle of non-positive weight passing through e in $D(S \cup E)$ if and only if such a cycle exists in $D(S' \cup E)$. But for every simple path p from x to y, $w_{S'}(p) = w_S(p) - k\epsilon$, where $k < n$ and $w_S(p)$ is an integer. Hence, $\lceil w_{S'}(p) \rceil = w_S(p)$, and since C is integral, $w_{S'}(p \cup e) \leq 0$ if and only if $w_S(p \cup e) \leq 0$. □

By lemma 3.1, and lemma 3.2, solving an instance of $DCIG$ linearly reduces into determining if at least one of two systems of difference constraints is feasible. Using the distance graph reformulation, the feasibility of such a system with M weak inequalities on N variables, with sum of absolute values of arc weights C, can be decided in $O(\min(NM, \sqrt{N}M \log NC))$ time [30, 13]. In our instance (G, A) there are n vertices, so $N = 2n, M = \Theta(n^2)$. Hence:

COROLLARY 3.5. *Deciding if a UCO graph with difference constraints has a realization can be done in $O(\min(n^3, n^{2\frac{1}{2}} \log nC))$ time.* ■

Note that the algorithms of [30, 13] for deciding the feasibility of a system also produce a feasible solution if one exists. This enables construction of a realization (if one exists) in $O(\min(n^3, n^{2\frac{1}{2}} \log nC))$ time.

3.2. Reducing a System of Difference Constraints to *BIG* on UCO graphs. Given a system of weak difference constraints, we shall show how to reduce it, in linear time, to an equivalent instance of BIG, in which the graph is UCO. According to lemma 3.2, the assumption that all constraints are weak can be made without loss of generality.

Let P be the following system of weak difference constraints in the variables $X = \{x_1, \ldots, x_N\}$:

(P) $$x_{j_i} - x_{k_i} \leq c_i \qquad i = 1, \ldots, M$$

Fix $C > 1 + \sum_{i=1}^{M} |c_i|$, and let $c'_i = c_i + (j_i - k_i)C$. Define a new system P' of difference constraints on the same variable set X:

(P') $$x_{j_i} - x_{k_i} \leq c'_i \qquad i = 1, \ldots, M$$

Note that the choice of C guarantees that $c'_i > 1$ (< -1) if and only if $j_i > k_i$ $(j_i < k_i)$, so P' can be rewritten as:

(P') $$\begin{array}{ll} x_{j_i} - x_{k_i} \leq c'_i & j_i > k_i \\ x_{k_i} - x_{j_i} \geq -c'_i & j_i < k_i \end{array}$$

where all right hand side terms are larger than one.

We call a solution $\{\tilde{x}_i\}_{i=1}^{N}$ to P' *monotone* if for each $i = 1, \ldots, N-1$: $\tilde{x}_i < \tilde{x}_{i+1} - 1$.

Lemma 3.6. *P has a solution if and only if P' has a monotone solution. Moreover, if $\bar{X} = \{\bar{x}_i\}$ is a feasible solution to P for which $\Delta = \max\{\bar{x}_i\} - \min\{\bar{x}_i\}$ is minimal, then $\bar{X}' = \{\bar{x}'_i | \bar{x}'_i = \bar{x}_i + iC\}$ is a monotone feasible solution to P'.*

Proof. Suppose P' has a monotone solution $x'_1 < x'_2 < \cdots < x'_N$. Let $\tilde{x}_i = x'_i - iC$, for each $1 \leq i \leq N$. By P' we get, for each $1 \leq i \leq M$: $\tilde{x}_{j_i} - \tilde{x}_{k_i} = x'_{j_i} - x'_{k_i} - (j_i - k_i)C \leq c'_i - (j_i - k_i)C = c_i$. Therefore, the i-th inequality in P is satisfied by $\{\tilde{x}_i\}_{i=1}^{N}$. Hence, P has a feasible solution.

Let $\tilde{x}_1, \ldots, \tilde{x}_N$ be a solution of P, for which $\Delta = \max\{\tilde{x}_i\} - \min\{\tilde{x}_i\}$ is minimal. (P defines an intersection of closed half-spaces, which is a closed set, therefore there is a solution attaining this minimal value). By [5], Δ is the sum of arc weights along some simple path in the distance graph $D(P)$, hence, $\Delta < C - 1$. Let $x'_i = \tilde{x}_i + iC$, for each $1 \leq i \leq N$. By P we get, for each $1 \leq i \leq M$: $x'_{j_i} - x'_{k_i} = \tilde{x}_{j_i} - \tilde{x}_{k_i} + (j_i - k_i)C \leq c_i + (j_i - k_i)C = c'_i$. Hence, $X' = \{x'_i\}_{i=1}^{k}$ is a feasible solution of P'. For each $1 \leq i < j \leq N$: $\tilde{x}_i - \tilde{x}_j \leq \Delta < (C-1)(j-i)$, hence, $x'_i - x'_j = \tilde{x}_i - \tilde{x}_j + (i-j)C < -1$ and X' is monotone. □

For the above system P, define $J = (G, U, L)$ to be the following BIG instance (compare figure 1):

- G is the intersection graph of the set of intervals $A \cup B \cup W$, defined as follows:

- $A = \{a_i\}_{i=0}^{N}$ where $a_i = [i, i+1]$.
- $B = \{b_{i/2}\}_{i=1}^{2N+1}$ where $b_x = [x, x]$.
- $W = \{w_{j_i k_i}\}_{i=1}^{M}$ where if $j_i > k_i$ then $w_{j_i k_i} = [k_i, j_i]$, and if $j_i < k_i$ then $w_{j_i k_i} = [j_i + \frac{1}{4}, k_i - \frac{1}{4}]$.

- The length constraints are as follows:
  - $U(a_i) = \infty$, $L(a_i) = 0$.
  - For integral i: $U(b_i) = L(b_i) = 0$, and $U(b_{i+\frac{1}{2}}) = L(b_{i+\frac{1}{2}}) = 1$,
  - If $j_i > k_i$: $L(w_{j_i k_i}) = 0$, $U(w_{j_i k_i}) = c'_i$.
  - If $j_i < k_i$: $L(w_{j_i k_i}) = -c'_i - \epsilon$, where $\epsilon < 1/N$, and $U(w_{j_i k_i}) = \infty$.

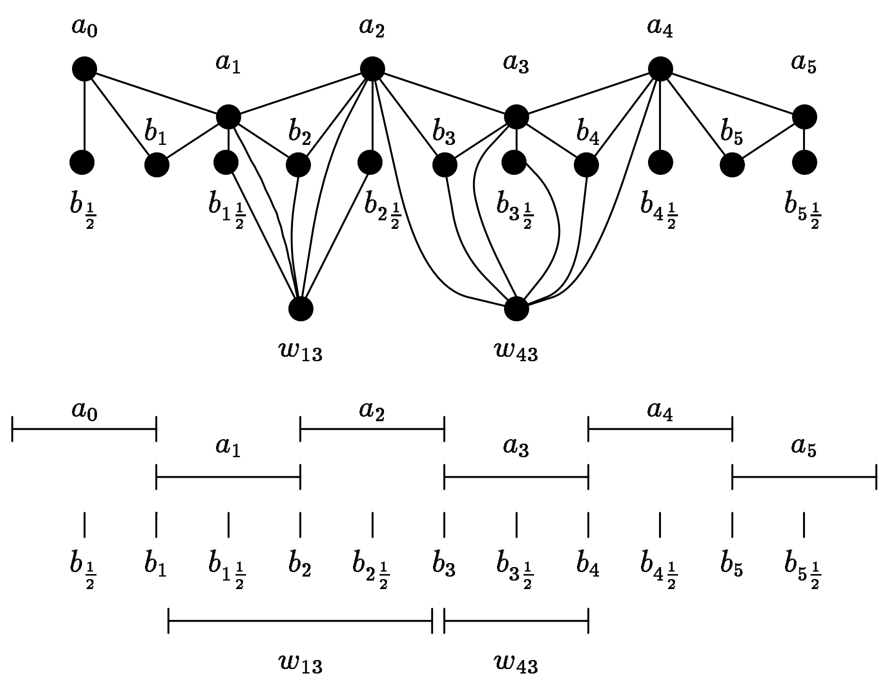

FIG. 1. *The graph G used in the reduction (top) and a realization for it (bottom)*

LEMMA 3.7. *G is UCO.*

Proof. Let G' be the intersection graph of $A \cup B$. It is easy to see that G' is prime, and hence, it has a unique clique order [34]. Moreover, G' has exactly $2N+1$ maximal cliques, each one containing (among other vertices) a unique and distinct b_i. The set of maximal cliques in G is $\{N[b_x]\}_{b_x \in B}$, namely, each clique is distinguished by a single b_i. Since G' is UCO, its unique clique order determines a unique linear order on $\{b_x | x \in B\}$, and hence, also on the maximal cliques of G. Hence, G is UCO. ▯

THEOREM 3.8. *P has a feasible solution if and only if J has a realization.*

Proof. **Only if:** Suppose $\{\tilde{x}_i\}_{i=1}^{N}$ is a feasible solution to P, for which $\Delta = \max\{\tilde{x}_i\} - \min\{\tilde{x}_i\}$ is minimal. By lemma 3.6, $\{x'_i\}_{i=1}^{N}$, where $x'_i = \tilde{x}_i + iC$ is a monotone solution to P'. Choose arbitrary

$x'_0 < x'_1 - 1$, $x'_{N+1} > x'_N + 1$. Define the following set $R \cup T \cup S$ of intervals:

- $T = \{T_i\}_{i=0}^N$ where $T_i = [x'_i, x'_{i+1}]$.
- $R = \{R_{i/2}\}_{i=1}^{2N+1}$ where $R_i = [x'_i, x'_i]$ if i is integral, and $R_{i+\frac{1}{2}} = [\frac{x'_i + x'_{i+1} - 1}{2}, \frac{x'_i + x'_{i+1} + 1}{2}]$ otherwise.
- $S = \{S_i\}_{i=1}^M$ where $S_i = [x'_{k_i}, x'_{j_i}]$ if $j_i > k_i$, and $S_i = [x'_{j_i} + \frac{\epsilon}{2}, x'_{k_i} - \frac{\epsilon}{2}]$ if $j_i < k_i$.

Since X' is monotone, the intersection graph of $R \cup T \cup S$ is isomorphic to G. The length bounds on vertices of T and R are trivially satisfied. If $j_i > k_i$ then $|S_i| = x'_{j_i} - x'_{k_i} \le c'_i$ as required. If $j_i < k_i$ then $x'_{k_i} - x'_{j_i} \ge -c'_i$, so indeed $|S_i| = x'_{k_i} - x'_{j_i} - \epsilon \ge -c'_i - \epsilon$, satisfying the length bounds on the vertices of S.

If: Suppose J has a realization. Let $\{y_i\}_{i=1}^N$ be the points in a realization of J which correspond to the intervals $\{b_i\}_{i=1}^N$ (which have length zero). W.l.o.g. $y_N > y_1$, because otherwise we can reverse the realization. Since G is UCO, the order of the intervals in J is identical to the order of the intervals $A \cup B \cup W$ in the definition of G. Therefore $y_i < y_j$ if and only if $i < j$, and due to the length constraint on $b_{i+\frac{1}{2}}$: $y_i < y_{i+1} - 1$, for $i = 1, \ldots, N-1$. Let S_i be the interval corresponding to $w_{j_i k_i}$ in the realization. Define a system P'' of difference constraints as follows:

(P")
$$\begin{array}{ll} x_{j_i} - x_{k_i} \le c'_i & j_i > k_i \\ x_{k_i} - x_{j_i} > -c'_i - \epsilon & j_i < k_i \end{array}$$

If $j_i < k_i$ then $y_{k_i} - y_{j_i} > |S_i| \ge -c'_i - \epsilon$, and if $j_i > k_i$ then $y_{j_i} - y_{k_i} \le |S_i| \le c'_i$. It follows that $\{y_i\}_{i=1}^N$ is a monotone solution to P''. A proof similar to lemma 3.2 implies that P' and P'' are equivalent, so P' is feasible. We would like to show that P' has a monotone solution. Let Q' be the system of constraints $x_i < x_{i+1} - 1$, $i = 1, \ldots, N-1$. $P' \cup Q'$ and $P'' \cup Q'$ have only monotone solutions. According to lemma 3.4, adding Q' to both P' and P'' maintains the equivalence between them. But a monotone solution of P'' realizes $P'' \cup Q'$, hence, P' has a monotone solution and according to lemma 3.6 P is feasible. ◻

COROLLARY 3.9. *The problem of deciding whether there exists a feasible solution to a system of difference constraints is linearly reducible to the problem BIG on a UCO graph.*

4. *DCIG* is NP-complete. We will now show, that although $DCIG$ is polynomial when restricted to UCO graphs, it is NP-complete in general. A stronger result will be proven in the next section, but we include a sketch of this proof as it is much more transparent.

THEOREM 4.1.

DCIG is strongly NP-complete.

Proof. We show a pseudo-polynomial reduction from the problem 3-PARTITION which is known to be strongly NP-complete (see, e.g., [15]).

An instance of 3-PARTITION is a set X of $n = 3k$ real numbers $x_1, \ldots, x_n \in (\frac{1}{4}, \frac{1}{2})$, such that $\sum_{i=1}^n x_i = k$. The question is to determine whether there exists a partition of X into k subsets (which have to be triplets) $X_1, \ldots, X_k$ so that for each $1 \le j \le k$: $\sum_{x \in X_j} x = 1$?

Let $X = \{x_1, \ldots, x_n\}$ be an instance of 3-PARTITION. Define an instance of $DCIG$, $I = (G, S)$

where G is the empty graph on the vertices $\{v_j\}_{j=1}^n \cup \{a_j\}_{j=0}^k$, and S consists of the following three types of constraints:

- $r(v_j) - l(v_j) \geq x_j$, for each $1 \leq j \leq n$.
- $l(a_{j+1}) - r(a_j) = 1$, for each $0 \leq j \leq k-1$.
- $r(a_0) \leq r(v_j) \leq l(a_k)$, for each $1 \leq j \leq n$.

We shall see that I is satisfiable if and only if X is a "yes" instance (see figure 2). Assume for now that all intervals in X must be open.

Suppose there exists a partition $X_1, \ldots, X_k$ as required, where $X_j = \{x_j^i\}_{i=1}^3$. Define $s_j^i = \sum_{r=1}^i x_j^r$, $s_j^0 = 0$. Examine the set of intervals $T = \{I_{a_j}\}_{j=0}^k \cup \{I_{v_j^i}\}_{1\leq j\leq k, 1\leq i\leq 3}$ where $I_{a_j} = (2j-1, 2j)$ and $I_{v_j^i} = (2j + s_j^{i-1}, 2j + s_j^i)$. The intervals in T are disjoint, and their endpoints trivially satisfy S, hence, T is a realization of I.

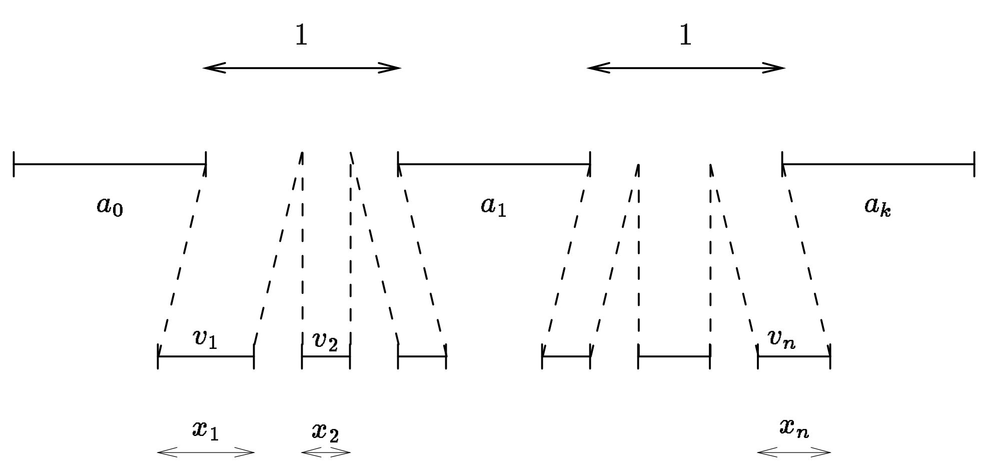

FIG. 2. *The v_i's can be squeezed between the a_i's if and only if a 3-partition exists*

Conversely, suppose $\{I_{a_i}\}_{i=0}^k \cup \{I_{v_i}\}_{i=1}^n$ is a realization of I. For each $1 \leq j \leq k$ define $I_j = (r(a_{j-1}), l(a_j))$, $X_j = \{x_i | I_{v_i} \subseteq I_j\}$. According to the constraints, $l(a_0) < r(a_0) < \cdots < l(a_k) < r(a_k)$, the I_j's do not intersect each other, and therefore the sets X_j are disjoint. Moreover, every x_i is a member of some X_j. Therefore $X_1, \ldots, X_k$ is a partition of X. For each $1 \leq j \leq n$: Since G is empty all the I_{v_i}'s are disjoint, hence, $\sum_{x_i \in X_j} x_i \leq |I_j| = 1$, hence, $X_1, \ldots, X_k$ form a 3-partition.

We assumed here that all intervals in the realization are open. To form a closed realization, it suffices to modify the reduction by allowing an interval of length $1 + \epsilon$ (instead of length 1) for each 'gap' interval $[r(a_{i-1}), l(a_i)]$, where ϵ is sufficiently small. (If each $a_i = \frac{p_i}{q_i}$, where p_i, q_i are integers, then $\epsilon < \frac{1}{4}(\max_i q_i)^{-3}$ suffices).

Since 3-partition is strongly NP-complete, and the reduction is pseudo-polynomial, our problem is strongly NP-complete. $\square$

5. Recognizing Measured Interval Graphs is NP-Complete. In this section we prove the NP-completeness of the problem MIG^*, introduced in section 1. The main part in this proof is a hardness result for the following, slightly more general problem, in which we specify in advance for each interval whether it should be closed or open:

RECOGNIZING A MEASURED INTERVAL GRAPH WITH SPECIFIED ENDPOINTS (MIG):
INSTANCE: A graph $G = (V, E)$, a non-negative *length* $L(v)$ for every $v \in V$, and a function $\phi : V \to \{open, closed\}$.
QUESTION: Is there a realization of G, in which the length of I_v is exactly $L(v)$, and I_v is open if and only if $\phi(v) = open$?

We shall denote such an instance by $P = (G, L, \phi)$. When P is a "yes" instance, we say that P is a *measured interval graph* (with endpoint specification). We shall first prove that MIG is NP-complete, and then reduce MIG to MIG^*.

The issue of endpoint specification seems unnatural at first sight. It is well known that for interval graphs in general the endpoint specification can be arbitrary, namely, a graph is interval if and only if it has a realization for any possible specification of endpoints. This is not the case in the presence of length constraints. For example: a $K_{1,3}$ graph with length 1 assigned to all vertices has no realization if all intervals are open (or all closed), but it has a realization precisely if the degree-3 vertex and two of the others are closed, as in figure 3.

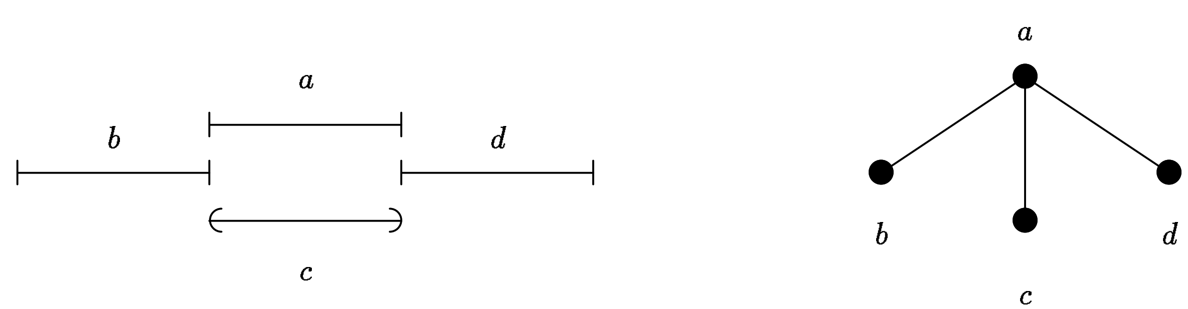

FIG. 3. *The $K_{1,3}$ graph shown (right) has a realization (left) if all intervals but c are closed.*

We shall often use the following implicit formulation for the problem, by representing G and ϕ using intervals, and using L to modify their length:

MIG: IMPLICIT FORMULATION:
INSTANCE: A pair (T, L) where $T = \{I_x\}_{x \in V}$ is a set of intervals, and $L : V \to \mathbb{Q}^+$ is a length function.
QUESTION: Is there a set of intervals $S = \{J_x\}_{x \in V}$ s.t.: (i) $J_x \cap J_y \neq \phi$ if and only if $I_x \cap I_y \neq \phi$. (ii) $|J_x| = L(x)$ for all x. (iii) J_x is closed if and only if I_x is closed.

This formulation is sometimes more convenient as it suggests a possible realization. We need the following notations and definitions:

DEFINITION 5.1.

Let $P = (G, L, \phi)$ be a measured interval graph with endpoints specification, and let $U \subseteq V$ be a set of its vertices. Define the measured interval graph P_U induced *by P on U, to be (G_U, L_U, ϕ_U), where G_U is the subgraph of G induced on U, and L_U, ϕ_U are the restrictions of L and ϕ, respectively, to U.*

Call two MIG instances $P = (G, L, \phi)$ and $P' = (G', L', \phi')$ isomorphic *if there is a graph isomorphism f between G and G', and for each $v \in V(G)$: $L(v) = L'(f(v))$, and $\phi(v) = \phi'(f(v))$, namely, the length and closure properties of intervals are preserved by f. In this case denote $P \cong P'$.*

DEFINITION 5.2.

Let $P = (G, L, \phi)$ be an instance of MIG. Let $S = \{I_1, \ldots, I_{|V(G)|}\}$ be a realization of P. Define $Length(S) = \sup\{x \in I | I \in S\} - \inf\{x \in I | I \in S\}$. Define $Length(P) = \inf\{Length(S) | S$ is a realization of $P\}$.

5.1. Basic Structures. We now describe three "gadgets" which are building blocks in our NP-completeness construction, and prove some of their properties. The structure of these gadgets assures us that their realization has very few degrees of freedom. To formalize this we introduce the following notion:

DEFINITION 5.3. *Two realizations of the same interval graph are* isometric *if they are identical up to reversal and an additive shift. Namely, there exists a function $f(x) = s \cdot x + c$ where $s = \pm 1$ and $c \in \mathbb{R}$, and $f(I_j) = I'_j$ for all j. Let $P = (G, L, \phi)$ be an instance of MIG. We call $U \subseteq V(G)$* rigid in *P if in any two realizations of P, the sets of intervals realizing U are isometric. In particular, all endpoints are located at fixed distances from the leftmost endpoint, including the rightmost one. Thus in every realization U has the same length. If $V(G)$ is rigid in P, we call P* rigid.

Note that the fact that U is rigid in P does not imply that P_U is rigid. For example, the instance P defined implicitly by the intervals in figure 3 is rigid, and in particular $\{b, c, d\}$ is rigid in P, but $P_{\{b,c,d\}}$ is not rigid.

5.1.1. The Switch. We first define the switch, a gadget which will be used as a toggle in larger structures. For the parameter real value $a \geq 1$, define the MIG instance *Switch(a)* $= (G, L, \phi)$ as follows (compare figure 4): G is the graph on the five vertices v_1, v_2, v_3, v_4, v_5, with edges $v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5$. L assigns lengths $0, \frac{1}{2}, 1, \frac{1}{2}, a - 1$ to $v_1, \ldots, v_5$, respectively, and $\phi(v)$ specifies v_3 to be *open*, and all the other vertices to be *closed*.

A realization $\{I_1, I_2, I_3, I_4, I_5\}$ of a $Switch(a)$ will be called *straight* if I_1 is to the left of I_5. Otherwise, it will be called *reversed*. We say that such a realization is *located* at I_3. For a straight realization U of a $Switch(a)$ located at $(x, x + 1)$, denote by $-U$ the reverse realization located at $(x + a - 1, x + a)$. Hence, $-U$ is a "mirror image" of U, covering the same interval $[x, x + a]$ along the real line.

LEMMA 5.4. *$Switch(a)$ is rigid. In particular, $Length(Switch(a)) = a$.*

Proof. Let S be a straight realization, as in the top left of figure 4. Suppose S' is another realization, such that both leftmost endpoints, I_1 and I'_1 are identical. The intersection graph of a $Switch(a)$ is prime,

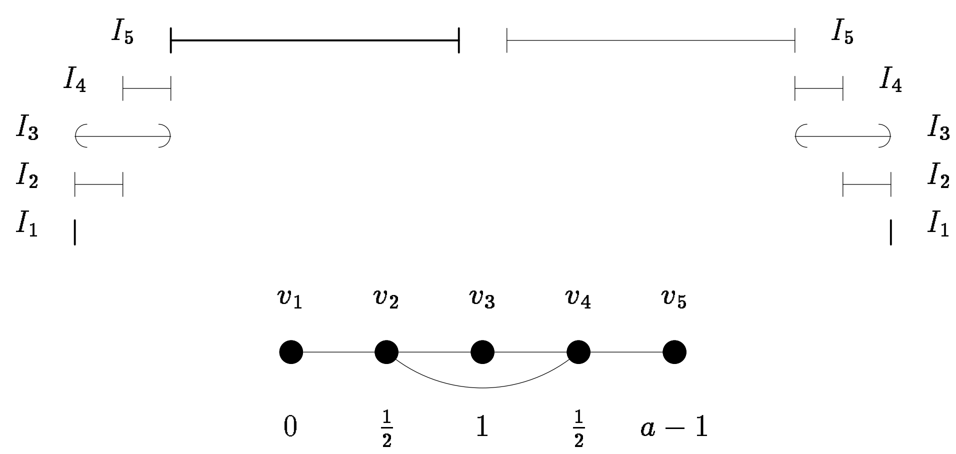

FIG. 4. *The* Switch *(bottom), a straight realization (top left) and a reversed realization(top right).*

hence, I_3 is between I_1 and I_5, and $l(I_5) \geq l(I_1)+1$. But $l(I_5) \leq l(I_4)+\frac{1}{2} \leq l(I_2)+\frac{1}{2}+\frac{1}{2} \leq l(I_1)+\frac{1}{2}+\frac{1}{2}$, therefore all inequalities hold as equalities. In particular, $Length(S) = a$, yielding $Length(Switch(a)) = a$. ∎

Note that lemma 5.4 implies that a realization of a straight $Switch(a)$ located at $(x, x+1)$ is unique. The same is true for a reversed *Switch*.

LEMMA 5.5.

Let $P = (G, L, \phi)$ be an instance of MIG, and let $S = \{I(v)\}_{v \in V(G)}$ be a realization for it. Let $U \subseteq V$ be a module such that $P_U \cong Switch(a)$. Then for $x, y \in N(U)$: $|I(x) \cap I(y)| \geq 1$.

Proof. Let $U = \{v_i\}_{i=1}^{5}$, with vertices numbered in the same order as in the definition of a *Switch*. According to lemma 5.4, $I(v_1)$ and $I(v_5)$ are one unit apart, but both of them intersect $I(x)$ and $I(y)$. Therefore $I(x) \cap I(y)$ contains the unit length interval between $I(v_1)$ and $I(v_5)$, yielding $|I(x) \cap I(y)| \geq 1$. ∎

5.1.2. The Fetters. Our second gadget binds two *Switch*es and imposes a prescribed distance between them. For positive real parameters, d, r_1, r_2, and two sets of vertices, U_1 and U_2, define a five-parameter instance of MIG, $Fetters(d, r_1, r_2, U_1, U_2)$, or in short, $Fetters = (G, L, \phi)$, as follows:

- U_1 and U_2 are modules in G, and each of them induces a *Switch*. More precisely, there exist constants a_1, a_2 such that $Fetters_{U_1} \cong Switch(a_1)$, $Fetters_{U_2} \cong Switch(a_2)$.
- The graph $\tilde{G} = (\tilde{V}, \tilde{E})$, constructed from G by contracting U_1, U_2 into vertices v_1, v_2 respectively, is as follows (compare figure 5):
  - $\tilde{V} = \{v^{tot}\} \cup \cup_{i=1,2}\{v_i, v_i^{end}, v_i^{short}, v_i^{long}\}$
  - $\tilde{E} = \hat{E} \cup E_1 \cup E_2$, where $\hat{E} = \{(v_2^{short}, v_1^{long}), (v_1^{short}, v_2^{long}), (v_1^{long}, v_2^{long}), (v_2^{long}, v_1), (v_1^{long}, v_2)\}$ and $E_i = \{(v_i^{short}, v_i^{long}), (v_i^{short}, v_i^{end}), (v_i^{end}, v_i^{long}), (v_i^{short}, v^{tot}), (v^{tot}, v_i^{long}), (v_i^{short}, v_i), (v_i, v_i^{long}), (v_i, v^{tot}), (v^{tot}, v_i)\}$ for $i = 1, 2$.

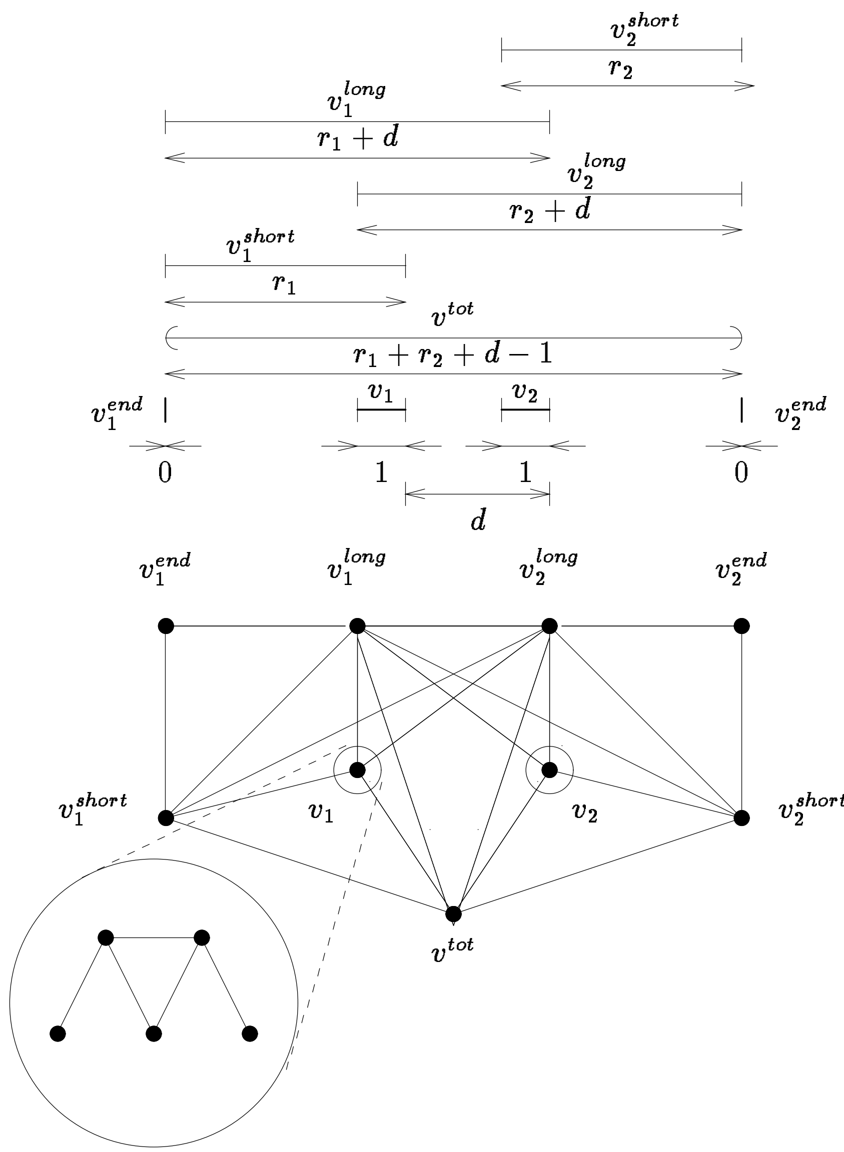

FIG. 5. *The Fetters (bottom) and a realization of it (top). The distance between the Switches v_1 and v_2 is fixed.*

- The lengths for the remaining intervals are:
  - $L(v^{tot}) = r_1 + r_2 + d - 1$.
  - $L(v_i^{short}) = r_i$ for $i = 1, 2$.
  - $L(v_i^{long}) = r_i + d$ for $i = 1, 2$.
  - $L(v_i^{end}) = 0$ for $i = 1, 2$.
- ϕ specifies v^{tot} to be *open*, and all the other intervals outside U_1, U_2 to be *closed*.

When there is no confusion, we shall use the vertex and the corresponding interval in the realization interchangeably. For example, $l(v_i^{short})$ is the position of the left endpoint of the interval corresponding to v_i^{short} in the realization, $|v_i^{short}|$ is its length, etc.

Call a realization of $Fetters$ *straight* if v_1 is to the left of v_2. Otherwise call the realization *reversed.* A realization of $Fetters$ is said to be *located* at the interval corresponding to v^{tot}. The $Fetters$ instance fixes the distance between its two $Switch$es. To formalize this notion we need the following definition:

DEFINITION 5.6. *Let $P = (G, L, \phi)$ be a MIG instance. Let $M, M' \subseteq V(G)$ be modules in G where $P_M \cong Switch(a)$, and $P_{M'} \cong Switch(a')$. For a realization of P, in which I and I' are the intervals corresponding to the middle vertices in M and M', respectively, define $Dist(M, M') = |l(I) - l(I')|$.*

LEMMA 5.7. *$\tilde{V} \setminus \{v_1, v_2\}$ is rigid in the $Fetters$. In particular, in every realization of the $Fetters$, $Dist(U_1, U_2) = d$.*

Proof. Recall that $\tilde{G}$ is the graph constructed from G by contracting U_1 and U_2 into v_1 and v_2, respectively. It is easy to see that $\tilde{G}$ is prime. Prime interval graphs have an interval order which is unique, up to complete reversal [34]. Hence, let us refer to the order in figure 5, where w.l.o.g. $l(v_1^{end}) = 0$, and $l(v_2^{end}) > 0$. v^{tot} is between v_1^{end} and v_2^{end}, so $l(v_2^{end}) \geq r_1 + r_2 + d - 1$. Furthermore, according to the length constraints: $l(v_2^{end}) \leq r_2 + l(v_2^{short})$, and $r(v_1^{long}) \leq r_1 + d$.

By lemma 5.5, $r(v_1^{long}) - l(v_2^{short}) = |v_1^{long} \cap v_2^{short}| \geq 1$, so $l(v_2^{end}) \leq r_2 + r(v_1^{long}) - 1 \leq r_1 + r_2 + d - 1$, yielding $l(v_2^{end}) = r_1 + r_2 + d - 1$, $v_1^{long} = [0, r_1 + d]$, $v_2^{short} = [r_1 + d - 1, r_1 + r_2 + d - 1]$, and $v_2 = [r_1 + d - 1, r_1 + d]$.

In a similar way we prove $v_1^{short} = [0, r_1]$, $v_2^{long} = [r_1 - 1, r_1 + r_2 + d - 1]$, and the result follows. ∎

By lemma 5.7, for a realization of the $Fetters(d, r_1, r_2, U_1, U_2)$ which is straight (or reversed) and located at $(x, x + r_1 + r_2 + d - 1)$, the only degrees of freedom are reversals of the $Switch$es.

5.1.3. The Frame. We now construct an element which divides an interval into sub-intervals of prescribed lengths. Each sub-interval is characterized by a distinct set of intervals which contain it. This element will be used as a frame, into which the moving and toggling elements will fit, and have the desired degrees of freedom.

Let $k = 2r - 1 \geq 3$, and let $x_1, x_2, \ldots, x_k$ be a sequence of real positive numbers, whose sum is s. Define $Frame(x_1, x_2, \ldots, x_k)$ to be an instance $P = (G(V, E), L, \phi)$ (see figure 6) as follows:

- $V = V^\alpha \cup V^\beta \cup V^\gamma$ consists of $3r+3$ vertices, where $V^\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $V^\beta = \{\beta_1, \beta_3, \ldots, \beta_k\}$, and $V^\gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$.
- The edges in G are:
  - $\gamma_2 v$ for each $v \in V$.
  - $\gamma_4 v$ for each $v \in V^\alpha \cup V^\beta$.
  - $\gamma_1 \beta_1$ and $\gamma_3 \beta_k$.
  - $\alpha_i \beta_j$ for each $1 \leq i, j \leq k$ such that j is odd and $|i - j| \leq 1$.
  - $\beta_i \beta_j$ for each $1 < i \neq j < k$ such that i, j are odd and $|i - j| \leq 2$.
- The lengths are:

- $L(\alpha_i) = x_i$ for each $1 \le i \le k$,
- $L(\beta_i) = x_{i-1} + x_i + x_{i+1}$ for each odd i, $3 \le i \le k-2$, and $L(\beta_1) = x_1 + x_2$, $L(\beta_k) = x_{k-1} + x_k$.
- $L(\gamma_1) = L(\gamma_3) = 0$ and $L(\gamma_2) = L(\gamma_4) = s$.

- ϕ specifies α_i to be *open* if i is odd, and γ_4 to be *open*, but all other intervals to be *closed*.

A realization of a $Frame$ is said to be *straight* if γ_1 is to the left of γ_3. Otherwise it is called *reversed*. Such a realization is *located* at the interval corresponding to γ_4.

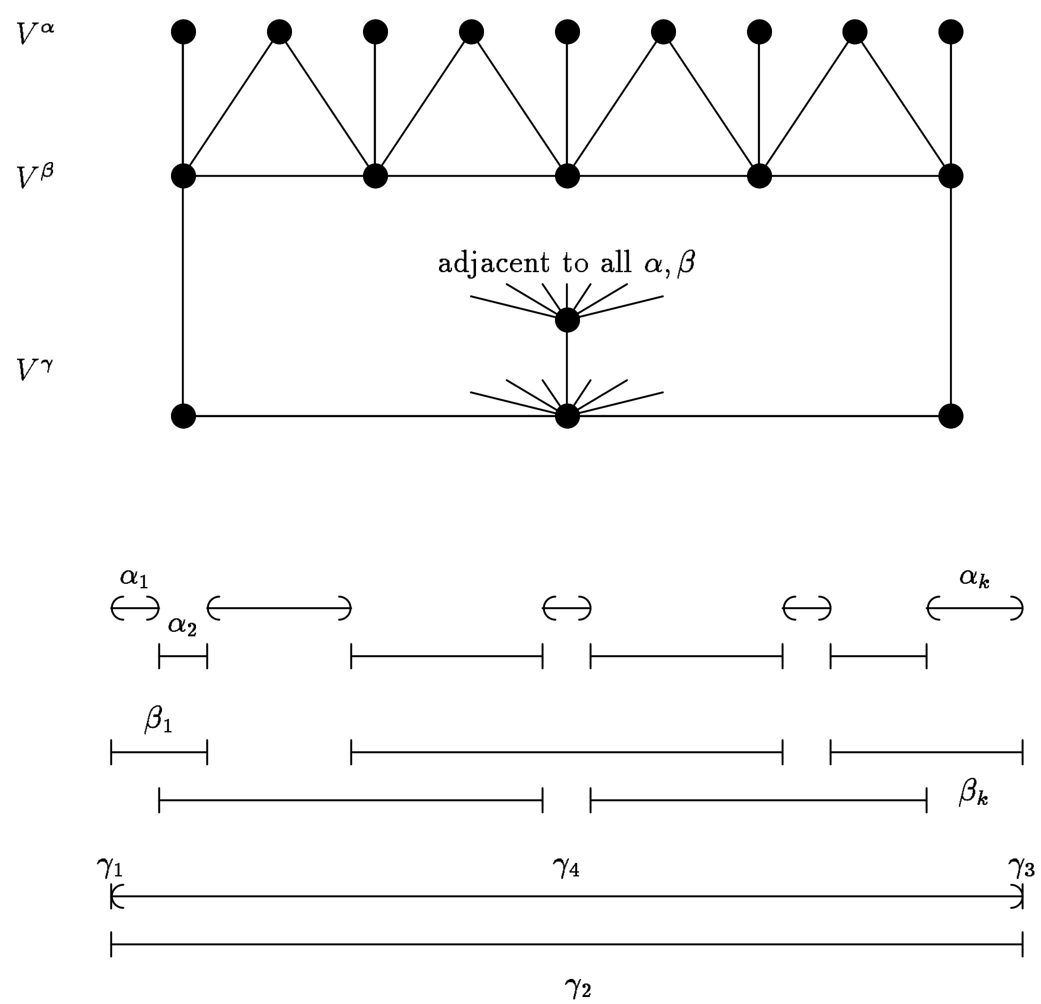

FIG. 6. *A graph of a Frame (top) and its realization (bottom). The Frame structure is rigid and divides an interval into smaller intervals of prescribed lengths and positions.*

LEMMA 5.8. *$V \setminus \{\beta_1, \beta_k\}$ is rigid in a $Frame$.*

Proof. Let G' be the subgraph of G induced on $V^\alpha \cup V^\beta$. It is easy to see that G' is prime, and hence, has a unique clique order [34]. Moreover, G' has exactly k maximal cliques, each one containing (among other vertices) a unique and distinct α_i. The set of maximal cliques in G is $\{N[\alpha_i]\}_{i=1}^{k}$, namely, each clique is distinguished by a single α_i. Since G' is UCO, its unique clique order determines a unique linear order on V^{α}, and hence, also on the maximal cliques of G. Hence, G is UCO.

Let S and S' be two realizations of the same $Frame$. Suppose $\gamma_1 = \gamma'_1$ are their leftmost endpoints, respectively. The $Frame$ graph is UCO, hence, the order of the α-intervals is identical in both S and S'. Moreover, $\sum_{i=1}^{k} L(\alpha_i) = s$, and all the α-intervals are disjoint, and must be between γ'_1 and γ'_3, which are at distance exactly s. Thus, the position of all α endpoints is uniquely determined. It is easy to see that also all β-intervals except β_1, β_k, must have identical position in both realizations. □

By lemma 5.8, for any straight (or reversed) realization of a $Frame(x_1, \ldots, x_k)$ located at $(x, x+s)$, the positions of all intervals except β_1, β_k, are uniquely determined.

In the sequel, when we use a realization of such a $Frame$ to implicitly define a MIG instance, we shall assume that β_1 and β_k are contained in $[x, x+s]$, so the realization has the shortest possible length. In addition, when we use any gadget in the implicit definition, and we describe its intervals by saying that "the gadget is located at ...", we mean that "a straight realization of the gadget is located at ...".

5.2. The Reduction. The realization of a MIG instance is a polynomial witness for a "yes" instance, hence, MIG is in NP. We describe a reduction from 3-Coloring, which is NP-complete (see, e.g., [15]). Let $G = (V, E)$ be an instance of 3-Coloring. We construct an instance $\mathbb{P} = (T, L)$ of MIG (in implicit form), and prove that $\mathbb{P}$ is a "yes" instance if and only if G is 3-colorable.

The general plan is as follows: We construct measured interval sub-instances for each vertex and for each edge of G. The sub-instance of a vertex is designed so that it can be realized only in three possible ways, which will correspond to its color. The sub-instance for each edge will prevent the vertices at its endpoints from having the same color.

5.2.1. The Vertex Sub-instance. Let $n = |V|$, $m = |E|$, $M = 24m + 11n + 1$. Define the following set $S = \omega \cup \delta \cup \zeta$ of intervals (compare figures 7 and 8):

- $\omega = \alpha \cup \beta \cup \gamma$ is a $Frame(1, 4, 1, 4, 1)$, located at $(0, 11)$.
- $\delta = \delta_1 \cup \delta_2$, where $\delta_j = \{\delta_j^i\}_{i=1}^{5}$, and each of δ_1 and δ_2 is a $Switch(3)$, located at $(1, 2)$ and $(7, 8)$, respectively. The superscripts match the vertex numbers in each $Switch$.
- $\zeta = \{\zeta^{tot}, \zeta_1^{end}, \zeta_1^{short}, \zeta_1^{long}, \zeta_2^{end}, \zeta_2^{short}, \zeta_2^{long}\}$, such that $\zeta \cup \delta_1 \cup \delta_2$ is a $Fetters(6, 2M+2, 2M-7, \delta_1, \delta_2)$ located at $(-2M, 2M)$.

Note that the intersection graph of $\omega \cup \zeta$ is prime.

For each interval $I \in S$ let $L(I) = |I|$. The sub-instance of each vertex is isomorphic to (S, L), and the $Frame$s of the n vertices are layed out contiguously as follows: For an interval J and a real number x, denote $J + x = \{y + x | y \in J\}$. Let $V = \{0, \ldots, n-1\}$. For each $i \in V$, $J \in S$ denote $J(i) = J + 11i$. Denote $S(i) = \{J(i)\}_{J \in S}$, and $\mathbb{S} = \cup_{i \in V} S(i)$. Let $\mathbb{P}^V$ be the measured interval graph defined implicitly by $\mathbb{S}$.

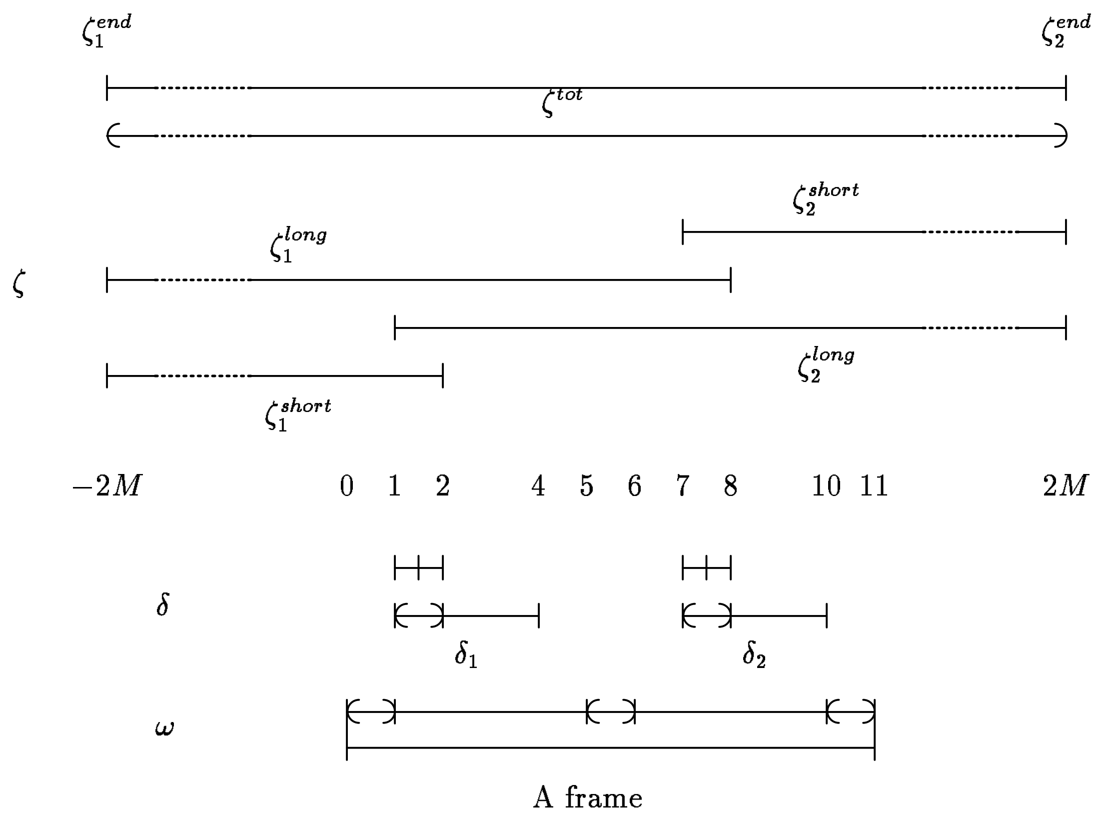

FIG. 7. *This is the set of intervals S. δ can be positioned in the frame. The distance between δ_1 and δ_2 is enforced by the ζ.*

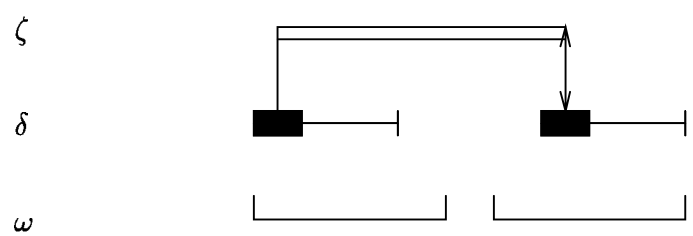

FIG. 8. *This is a sketch of the structure of a vertex. The Switches δ can be positioned in the frame ω. The distance between δ_1 and δ_2 is enforced by the ζ.*

A realization of a vertex sub-instance is called *straight* (respectively, reversed) if the realization of its ω is straight (respectively, reversed).

LEMMA 5.9. *Let $\cup_{i \in V} S(i)'$ be a realization of $\mathbb{P}^V$, with $S(i)' = \{J(i)'\}_{J(i) \in S(i)}$. Then either every $S(i)'$ is straight, or every $S(i)'$ is reversed.*

Proof. It suffices to prove that $l(\gamma_1(i)') < l(\gamma_3(i)')$ if and only if $l(\gamma_1(i+1)') < l(\gamma_3(i+1)')$. This follows from the identity of the zero-length intersecting intervals $\gamma_1(i+1)'$ and $\gamma_3(i)'$, and the disjointness of $\gamma_1(i)'$ and $\gamma_3(i+1)'$, which are both, by lemma 5.8, at distance 11 from the former pair, respectively. $\square$

Let $\mathbb{S}'$ be a straight realization of $\mathbb{P}^V$. Define the function $Col : V \longrightarrow R$, as follows:

$$Col(i) = l(\delta_1^3(i)') - l(\gamma_1(i)') - 1 \tag{2}$$

Call Col the *coloring* defined by $\mathbb{S}'$. We now show that each vertex subgraph can be realized in exactly three distinct colors. This is also demonstrated in figure 9.

LEMMA 5.10.

For each $i \in V$: $Col(i) \in \{0, 1, 2\}$.

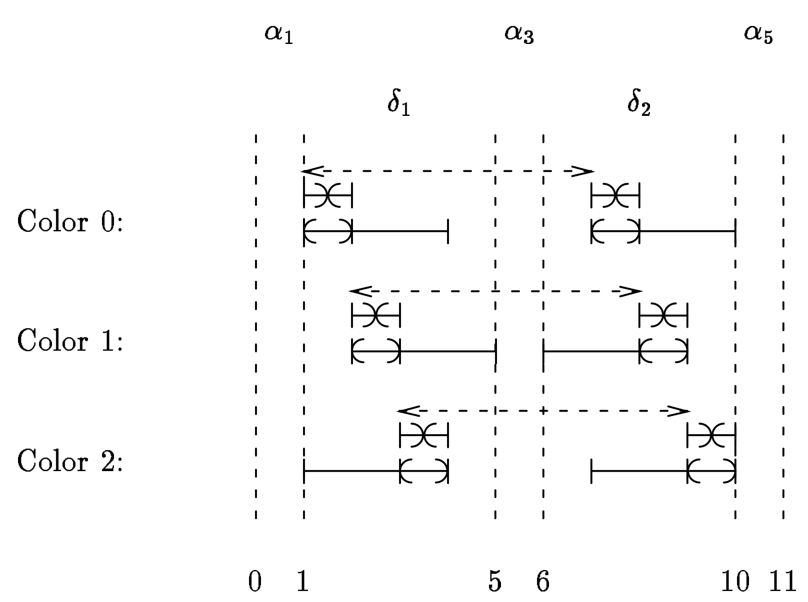

FIG. 9. *The three possible positions of δ_1,δ_2 in a vertex sub-instance, which will correspond to the three possible colors of the vertex.*

Proof. According to lemma 5.8, in each straight $S(i)'$, the positions of the intervals in $\alpha(i)'$ relative to $l(\gamma_1(i)')$, are fixed. Assume w.l.o.g. $l(\gamma_1(i)') = 0$. For each $J \in \delta_1(i)'$: $J \subseteq [1,5]$, and for each $J \in \delta_2(i)'$: $J \subseteq [6,10]$. By lemma 5.4, $Length(\delta_1(i)) = Length(\delta_2(i)) = 3$, hence (compare figure 9):

$$\text{if } \delta_1(i)' \text{ is straight, then} \quad 1 \leq l(\delta_1^3(i)') \leq 2 \tag{3}$$

$$\text{if } \delta_1(i)' \text{ is reversed, then} \quad 3 \leq l(\delta_1^3(i)') \leq 4 \tag{4}$$

$$\text{if } \delta_2(i)' \text{ is straight, then} \quad 6 \leq l(\delta_2^3(i)') \leq 7 \tag{5}$$

$$\text{if } \delta_2(i)' \text{ is reversed, then} \quad 8 \leq l(\delta_2^3(i)') \leq 9 \tag{6}$$

But according to lemma 5.7:

(7) $$l(\delta_1^3(i)') - l(\delta_2^3(i)') = 6$$

Therefore,

(8) $$l(\delta_1^3(i)') \in \{1, 2, 3\}$$

and $Col(i) \in \{0, 1, 2\}$. □

5.2.2. The Edge Sub-instance. Let the edges of G be $E = \{e_0, \dots, e_{m-1}\}$, and let $e_k = (i, j)$ be an edge in E, where $i < j$. For each edge e_k we construct an edge sub-instance, that forces the colors of the vertices i and j to be different.

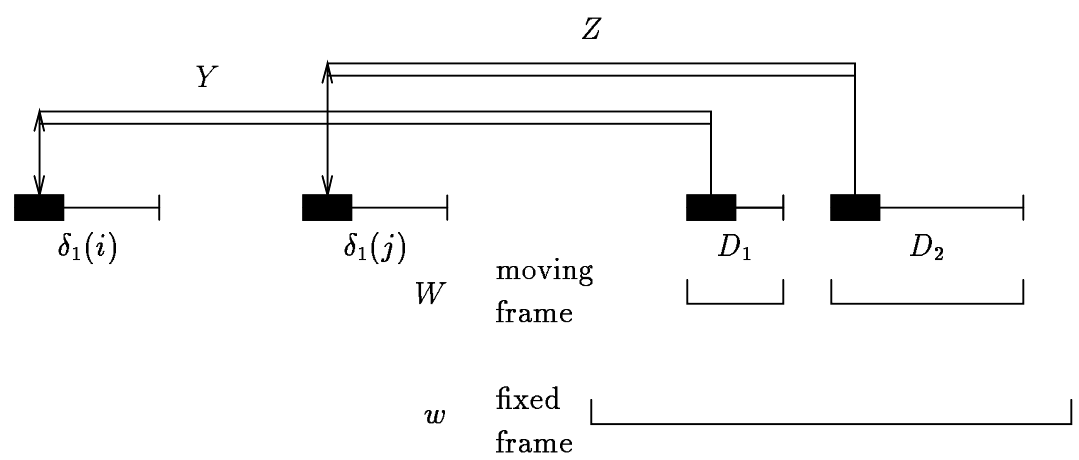

FIG. 10. *This is a functional sketch of the edge sub-instance. The two Switches D can only reverse in their relative fixed positions inside the moving frame. Their distances from the corresponding Switches in the vertices i, j, respectively, are fixed. The moving frame itself can have different positions along the fixed frame.*

We first give an overview of this construction (compare Figure 10): Each edge is assigned a fixed *Frame* w which contains two *Switch*es, D_1 and D_2, which are the heart of its sub-instance. The *Frame*s of the edges are layed out contiguously to the right of the vertex *Frame*s. The sub-instance is a collection of intervals $\{A_0\} \cup W \cup D \cup Y \cup Z \cup w$ designed so that:

1. D_1 and $\delta_1(i)$ are kept at a fixed distance (this is done by the Y intervals).
2. D_2 and $\delta_1(j)$ are kept at a fixed distance (this is done by Z).
3. D_1 and D_2 are restricted to be in one of four possible relative positions, allowing the four possible color differences between the vertices i and j (this is done by W).
4. D_1 and D_2 together can undergo a translation, allowing the six possible color combinations of the vertices i and j, as demonstrated in figure 12 (this is done by A_0 and w).

We now describe the construction in detail (compare Figure 11): Define the following set $X(k) = \{A_0(k)\} \cup W(k) \cup D(k) \cup Y(k) \cup Z(k) \cup w(k)$ of intervals. Let $Base(k) = 18k + 11n$. For readability, we omit the parameter k whenever possible.

- $w = a \cup b \cup c$ is a $Frame(5, 12, 1)$, located at $(0, 18) + Base(k)$.
- $A_0 = [3, 7] + Base(k)$.
- $W = A \cup B \cup C$ is a $Frame(1, 2, 1, 4, 1)$, located at $(7, 16) + Base(k)$.
- $D = D_1 \cup D_2$, where $D_j = \{D_j^i\}_{i=1}^5$, and:
  - D_1 is a $Switch(2)$ located at $(8, 9) + Base(k)$.
  - D_2 is a $Switch(4)$ located at $(11, 12) + Base(k)$.
- Define $d_Y(k, i) = Base(k) + 8 - (11i + 2) = 11(n - i) + 18k + 6$.
  $Y = \{Y^{tot}, Y_1^{end}, Y_1^{short}, Y_1^{long}, Y_2^{end}, Y_2^{short}, Y_2^{long}\}$, such that $Y \cup D_1 \cup \delta_1(i))$ is the $Fetters(d_Y(k, i) + 1, 6k + 5 + 11i + 2, M - 6k + 4 - (11n + 18k + 8), \delta_1(i), D_1)$ located at $(-6k - 5, M - 6k + 4)$.
- Define $d_Z(k, j) = Base(k) + 11 - (11j + 1) = 11(n - j) + 18k + 10$.
  $Z = \{Z^{tot}, Z_1^{end}, Z_1^{short}, Z_1^{long}, Z_2^{end}, Z_2^{short}, Z_2^{long}\}$, such that $Z \cup D_2 \cup \delta_1(j)$ is the $Fetters(d_Z(k, j), 6k + 3 + 11j + 1 + 1, M - 6k - (11n + 18k + 11), \delta_1(j), D_2)$ located at $(-6k - 3, M - 6k)$.

The length function on the sub-instance (X, L) is defined so that $L(I) = |I|$ for all intervals except the Y-s in which we set:

$$L(Y_1^{short}) = |Y_1^{short}| + 1 \qquad L(Y_2^{long}) = |Y_2^{long}| - 1$$

This change, together with the $+1$ in the first parameter of the $Fetters$ of Y, forces a $+1$ shift on the location of $\delta_1(i)$. This shift will be crucial in forcing the vertices i and j to have different colors.

Note that the intersection graph of $X(k) - D(k)$ is prime. Note also that the left and right ends of the $Fetters$ sub-instances Y and Z are positioned way beyond the contiguous $Frames$, in all edge sub-instances and vertex sub-instances. This allows every $Fetters$ to move independently, and no vertex or edge subgraph is a module.

Denote $\mathbb{X} = \cup_{e_i \in E} X(i)$. Let $\mathbb{X}' = \cup_{J \in \mathbb{X}} J'$ be a set of intervals with the same intersection graph as of $\mathbb{X}$, which satisfy the corrected length constraints, where $X(i)' = \{J(i)'\}_{J(i) \in X(i)}$. Call $X(i)'$ a *straight* (respectively, *reversed*) realization if the frame $w(i)'$ is straight (respectively, reversed). A proof similar to that of lemma 5.9 implies:

Lemma 5.11. *For each $1 \leq i < j \leq m$: $X(i)'$ is straight if and only if $X(j)'$ is straight.* ■

The complete constructed instance is $\mathbb{P} = (\mathbb{S} \cup \mathbb{X}, L)$, where the interval lengths L are implicit in each of the two types of subgraphs, and the only exception is the corrected length in the $Y(k)$-s. Due to this exception simple super-imposition of $\mathbb{S}$ and $\mathbb{X}$ does not give a realization.

Lemma 5.12. *If $\mathbb{S}' \cup \mathbb{X}'$ is a realization of $\mathbb{P}$ for which $\mathbb{S}'$ is straight, then $\mathbb{X}'$ is straight.*

Proof. Recall that $c_1(1)$ and $\gamma_3(n)$ are the leftmost and the rightmost zero length intervals in the leftmost edge sub-instance, and the rightmost vertex sub-instance, respectively. Suppose, to the contrary, that $\mathbb{X}'$ is not straight. The zero-length intersecting intervals $c_1(1)'$ and $\gamma_3(n)'$ must be identical. W.l.o.g. $c_1(1)' = \gamma_3(n)' = [0, 0]$. Then $c_2(1)' = [-18, 0]$, $\gamma_1(n)' = [-11, -11]$, and these two intervals intersect, in contrary to our constructed interval graph. □

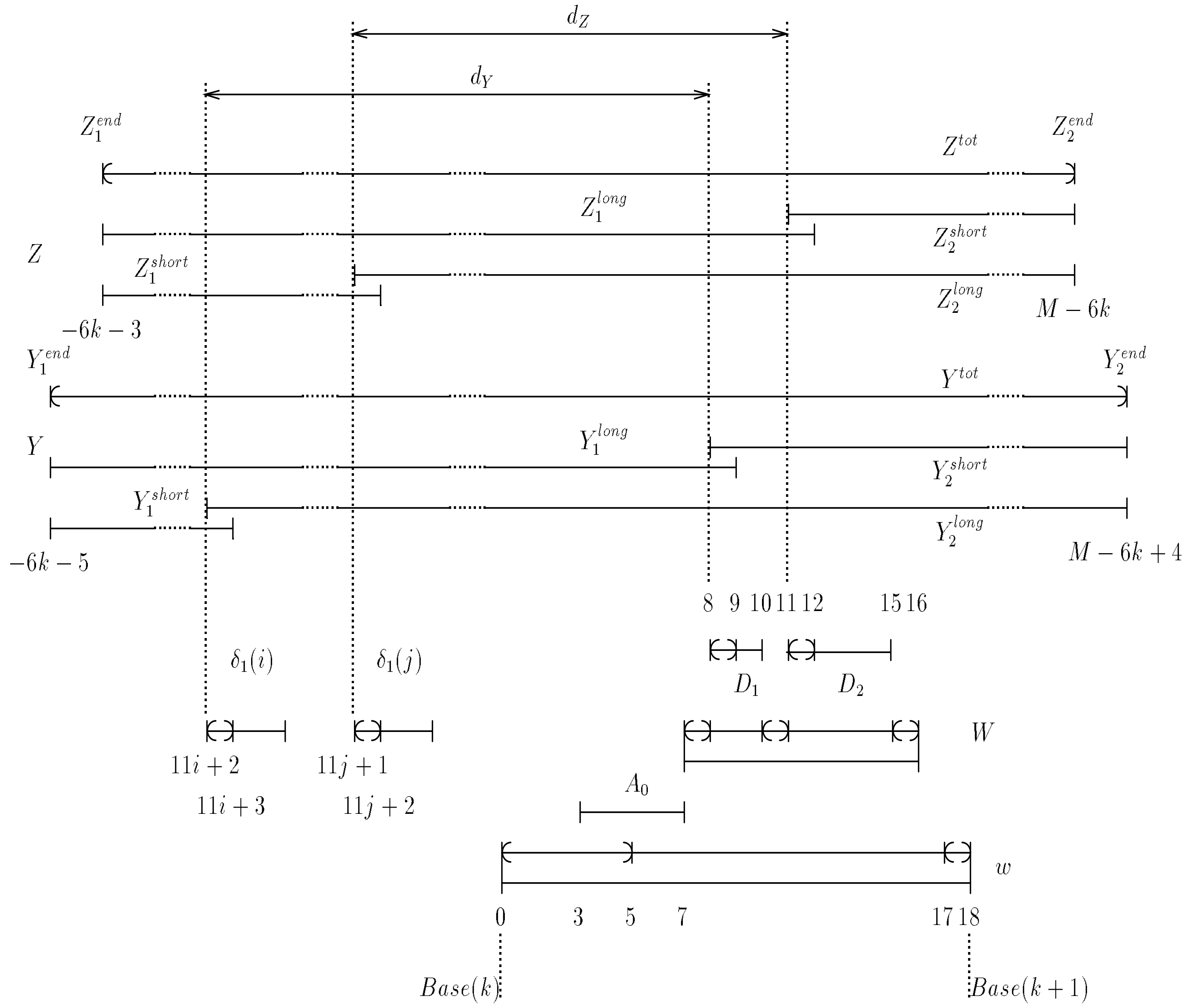

FIG. 11. *The edge sub-instance: A moving frame can be positioned inside the fixed frame. The* Switches *D_1 and D_2 are positioned inside the moving frame. Each of D_1 and D_2 is connected to its vertex sub-instance via Fetters.*

1

Y

FIG. 1. *The whole edge*

Lemma 5.13.

If $e_k = (i, j)$, then for every realization $\mathbb{S}' \cup \mathbb{X}'$ of $\mathbb{P}$: $Col(i) \neq Col(j)$.

Proof. Assume w.l.o.g that the realization is straight, and that $l(c_1(k)') = Base(k)$. Again, we omit the parameter k whenever possible. Surely

$$l(C_1{}') \leq r(A_0{}') < r(a_1{}') + 4 = 9 + Base(k) \tag{9}$$

The first inequality follows since C_1' and A_0' must intersect, the second inequality follows since A_0' and a_1' should intersect, and the last equality follows since w' is rigid (lemma 5.8). Since $Length(W) = |W'| = 9$, we conclude that $W' = A' \cup B' \cup C'$ is straight.

According to lemma 5.8 the relative positions of all the intervals in the $Frame$ (except B_1' and B_5') are fixed relative to $l(C_1')$. The realization for each of D_1 and D_2 can be either straight or reversed, giving rise to four possible combinations of positions (Any of these combinations fixes the positions of D' with respect to $l(C_1')$). In particular:

$$l(D_2^3{}') - l(D_1^3{}') \in \{2, 3, 5, 6\} \tag{10}$$

Due to lemma 5.7, and the realization being straight:

$$l(\delta_1^3(j)') - l(\delta_1^3(i)') \tag{11}$$

$$= \quad (l(D_2^3{}') - d_Z(k, j)) - (l(D_1^3{}') - d_Y(k, i)) \tag{12}$$

$$= (l(D_2^3{}') - (11(n-j) + 18k + 10)) - (l(D_1^3{}') - (11(n-i) + 18k + 6)) \tag{13}$$

$$= \quad l(D_1^3{}') - l(D_1^3{}') - 4 + 11(j-i) \in \{11(j-i) \pm 1, 11(j-i) \pm 2\} \tag{14}$$

Therefore:

$$Col(j) - Col(i) \tag{15}$$

$$= (l(\delta_1^3(j)') - l(\gamma_1(j))' - 1) - (l(\delta_1^3(i)') - l(\gamma_1(i))' - 1) \tag{16}$$

$$= \quad l(\delta_1^3(j)') - l(\delta_1^3(i)') - 11j + 11i \in \{\pm 1, \pm 2\} \tag{17}$$

□

Corollary 5.14.

If $\mathbb{P}$ is a "yes" instance, then G is 3-colorable.

Proof. If $\mathbb{P}$ is a measured interval graph, then it has a straight realization (since the realization can reversed completely). Define the coloring $Col : V \longrightarrow \{0, 1, 2\}$ as described in (2). By lemma 5.13, Col is a proper 3-Coloring of G. □

Let us now prove the converse:

Lemma 5.15.

If G is 3-colorable, then $\mathbb{P}$ admits a realization.

Proof. Let $Col : V \to \{0,1,2\}$ be a proper 3-coloring of G. We build a realization $\mathbb{S}' \cup \mathbb{X}'$ for the instance $\mathbb{P}$ as follows:

1. For the vertex $i \in V$ with $Col(i) = x$ position its *Switch*es δ_1', δ_2' as follows (compare figure 9):
   - If $x = 0$ then $\delta_1' = \delta_1$, $\delta_2' = \delta_2$.
   - If $x = 1$ then $\delta_1' = \delta_1 + 1$, $\delta_2' = -\delta_2 - 1$.
   - If $x = 2$ then $\delta_1' = -\delta_1$, $\delta_2' = -\delta_2$.

   The rest of the intervals in the vertex subgraph are positioned accordingly (cf. lemma 5.10).
2. For the edge $e_k = (i,j)$ with $y = Col(j) - Col(i)$, the directions of the *Switch*es $D_1(k)'$ and $D_2(k)'$ in the realization are determined by y, thus fixing the distance between $D_1^3(k)'$ and $D_1^3(k)'$. The absolute position of these *Switch*es is determined according to $Col(i)$ and $Col(j)$, as follows (compare figure 12):

$Col(i)$	$Col(j)$	$D_1(k)'$	$D_2(k)'$
0	1	$-D_1(k)' - 2$	$-D_2(k)' - 2$
0	2	$D_1(k)' - 1$	$-D_2(k)' - 1$
1	0	$D_1(k)'$	$D_2(k)'$
1	2	$-D_1(k)' - 1$	$-D_2(k)' - 1$
2	0	$-D_1(k)'$	$D_2(k)'$
2	1	$D_1(k)' + 1$	$D_2(k)' + 1$

$\mathbb{S}' \cup \mathbb{X}'$ and $\mathbb{S} \cup \mathbb{X}$ have the same intersection graph, all interval lengths match the prescribed lengths, and their endpoints meet the specification. □

From lemma 5.15 and corollary 5.14 we can finally conclude:

THEOREM 5.16. *MIG is NP-complete.* ■

In fact, the same reduction implies strong NP-completeness, as 3-Coloring is strongly NP-complete and the reduction is also pseudo-polynomial.

5.3. Closing the Open Intervals. We have proved that recognizing a measured interval graph with specified endpoints is NP-complete. We now show that this problem is hard even where all the intervals are closed. Given an instance $P = (G, L, \phi)$ of MIG, define a new instance $P' = (G, L')$ of MIG^* (in which all intervals are closed), as follows: Let $n = |V(G)|$, and fix $\epsilon < \frac{1}{20n^2}$.

$$L'(v) = \begin{cases} L(v) & \text{if } v \text{ is closed} \\ L(v) - 2\epsilon & \text{if } v \text{ is open} \end{cases} \tag{18}$$

Let $\mathbb{P}$ be an instance generated by the reduction in section 5.2. We shall prove that $\mathbb{P}$ has a realization if and only if $\mathbb{P}'$ has one.

First, we observe that the construction introduced in the proof of theorem 5.16 has a special property: Let S be a realization in which the shortest non-zero length of an interval is C. S is called *discrete* if all the endpoints of its intervals are integer multiples of C. In that case, C is called the *grid size* of S.

REMARK 5.17.

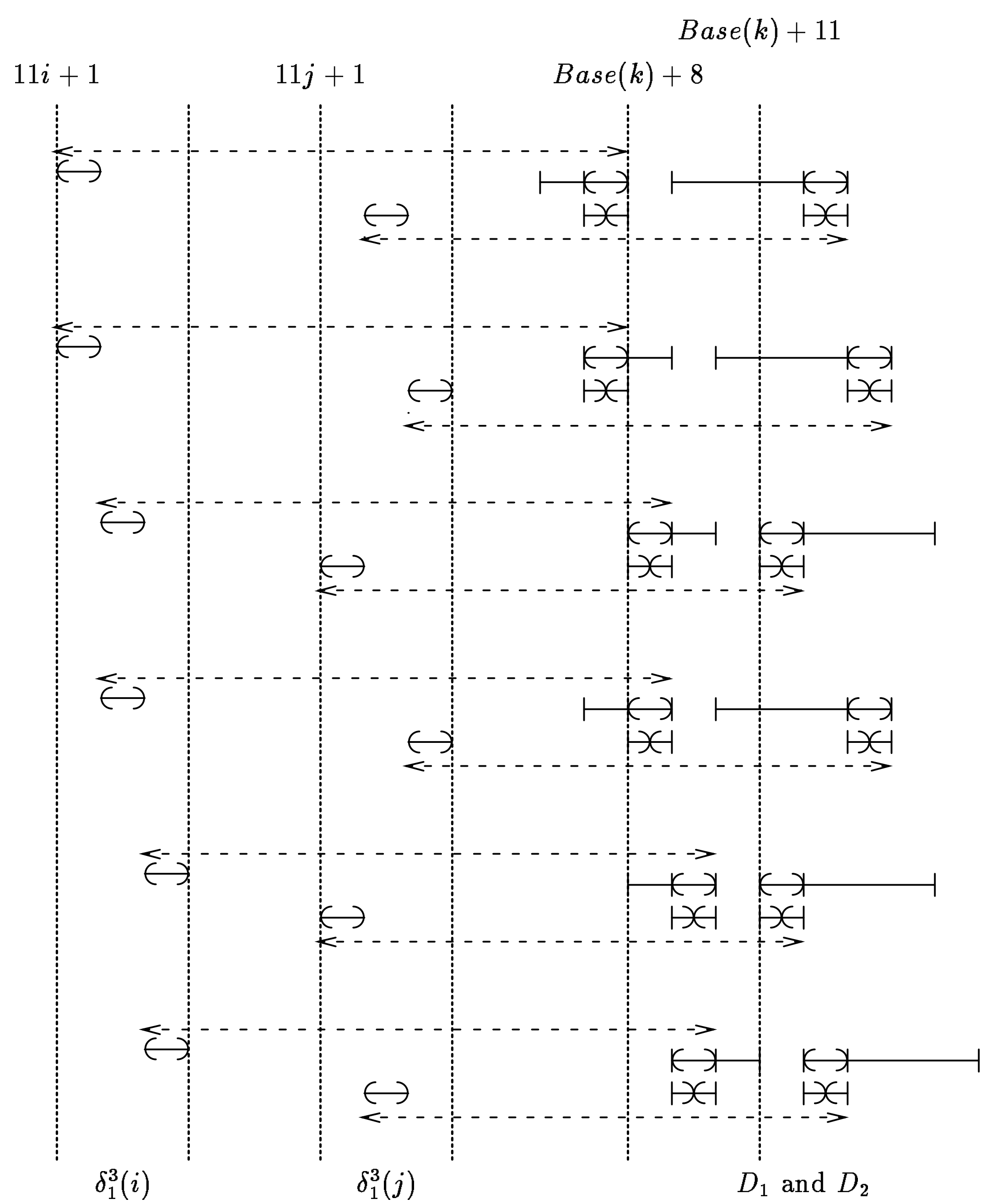

FIG. 12. *The relative position of D_1^3 and D_2^3 forces the colors of the vertices to be different.*

By the proofs of lemma 5.15 and corollary 5.14, $\mathbb{P}$ has a realization if and only if it has a discrete realization, with grid size $\frac{1}{2}$.

LEMMA 5.18.

If $\mathbb{P}$ has a realization, then $\mathbb{P}'$ has one.

Proof. If $\mathbb{P}$ has a realization, then by remark 5.17, it has a discrete realization $\{I_v\}_{v\in V(G)}$ with grid size $\frac{1}{2}$. Construct the set of closed intervals $\{I'_v\}_{v\in V(G)}$ defined as follows: If I_v is closed, let $I'_v = I_v$. If I_v is open, let $I'_v = [l(I_v)+\epsilon, r(I_v)-\epsilon]$.

We lemma that this set is a realization of $\mathbb{P}'$: Since $\mathbb{P}$ is discrete, the intervals I_v and I_u intersect if and only if I'_v and I'_u intersect, since if one (or both) of I_v, I_u is open, then their overlap is at least $\frac{1}{2}$. Furthermore, clearly $|I'_v| = L'(v)$ for every $v \in V$. □

Unfortunately, the converse of lemma 5.18 does not always hold for arbitrary MIG instances, as demonstrated in figure 13. We shall prove that the converse does hold for instances generated by the reduction in section 5.2.

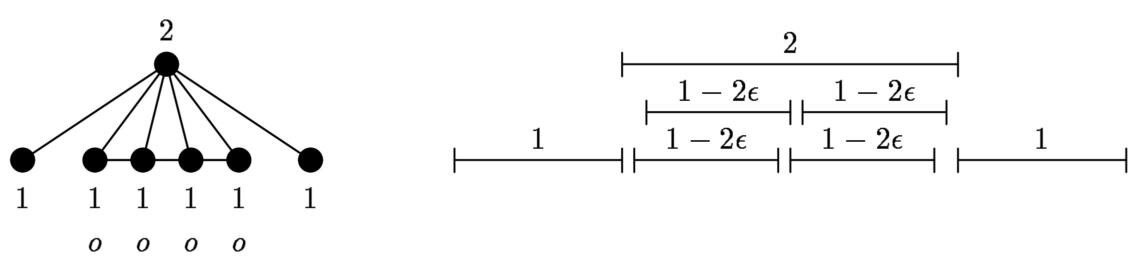

FIG. 13. *In the MIG instance P on the left, the the numbers denote lengths, and the four intervals corresponding to the vertices marked "o" should be open. P has no realization, but P′ has one, as shown on the right.*

Define the following order-oriented analog of MIG and MIG^*, respectively:

> RECOGNIZING A MEASURED INTERVAL ORDER WITH SPECIFIED ENDPOINTS (MIO):
> **INSTANCE:** A partial order $\prec$ on a set V, a non-negative *length* $L(v)$ for every $v \in V$, and a function $\phi : V \to \{open, closed\}$.
> **QUESTION:** Is there an interval realization of $(V, \prec)$, in which the length of I_v is exactly $L(v)$, and I_v is open if and only if $\phi(v) = open$?

MIO^* is the restriction of MIO to instances with all intervals closed. MIO^* can be solved in polynomial time [19, 24, 32], and that solution can be generalized to deal with open intervals and solve MIO as well.

We need to generalize the notion of rigidness in the following manner:

DEFINITION 5.19. *For a real $p \geq 0$, two realizations $\{I_j\}$ and $\{I'_j\}$ of the same interval graph are* p-isometric *if there exists a function $f(x) = s\cdot x + c$ where $s = \pm 1, c \in \mathbb{R}$, and constants c_j, $|c_j| < p$ such that $f(I_j) + c_j = I'_j$ for all j. We call $U \subseteq V(G)$* p-rigid *in a MIO instance if in any two realizations of the instance, the sets of intervals realizing U are p-isometric. Note that in this case all endpoints of*

U are located at fixed distances from the leftmost endpoint, up to $\pm p$. Hence, every realization has the same length, up to $\pm p$.

For an instance Q of MIO, define the following system of inequalities $S(Q)$, on the variables $\{l_v\}_{v \in V}$:

- If $x \prec y$, and both x, y are closed: $l_x + L(x) < l_y$.
- If $x \prec y$, and at least one of x, y is open: $l_x + L(x) \le l_y$.
- If $x \cap y \ne \phi$, and both x, y are closed: $l_x + L(x) \ge l_y$.
- If $x \cap y \ne \phi$, and at least one of x, y is open: $l_x + L(x) > l_y$.

Q has a realization if and only if $S(Q)$ has a feasible solution, since the left endpoints of the realization satisfy $S(Q)$, and vice versa. Recall that $D(S(Q))$ is the distance graph of $S(Q)$, as in the proof of lemma 3.2, and denote it $D(Q)$, for short.

Lemma 5.20. *Let Q be an instance of MIO. If U is a strongly connected component in the union of all zero-weight cycles in $D(Q)$, then U is rigid in Q.*

Proof. For vertices $x, y \in U$, there is a zero-weight cycle c in $D(Q)$ passing through both x and y. Let d (resp. $-d$) be the weight of the path from x to y (resp. y to x) along c. Summing the inequalities in $S(Q)$ along the two paths we get $l_y \le l_x + d$ and $l_x \le l_y - d$, respectively, implying $l_y - l_x = d$. Since, every realization must satisfy $S(Q)$, for every realization $l_y - l_x = d$, so U is rigid. □

The converse holds as well:

Lemma 5.21. *Let Q be a realizable instance of MIO, with U rigid in Q. Then for each $x, y \in U$ there is a zero-weight cycle in $D(Q)$ containing both x and y.*

Proof. Suppose to the contrary $x, y \in U$ and there is no zero-weight cycle in $D(Q)$ containing both x and y. Either there is a cycle in $D(Q)$ through x and y, or there is no such cycle.

If there is no cycle in $D(Q)$ through x and y, then w.l.o.g. there is no path in $D(Q)$ from x to y. Let $W \subseteq V \setminus \{x\}$ be the set of all vertices in V to which there is a path from y in $D(Q)$ (including y itself). Then Q does not contain any inequalities $v < w + C$ or $v \le w + C$ for $w \in W$, $v \in V \setminus W$. Let $\{I_v\}_{v \in V}$ be a realization for Q. Then $\{I_w - 1\}_{w \in W} \cup \{I_v\}_{v \in V \setminus W}$ also realizes Q, with a different distance between the intervals corresponding to x and to y, contradicting the rigidness of U in Q.

If there exists a cycle in $D(Q)$ through x and y, then let c be such a cycle of minimum weight, and let $l = w(c)$. By assumption $l \ne 0$, and since Q has a realization, by corollary 3.3 $l > 0$. Let d be the weight of the path from x to y along c. For every Δ, $d - l < \Delta < d$, consider a new directed graph $D'(\Delta)$ obtained from $D(Q)$ by adding two arcs xy and yx, both labeled $\le$, with weights Δ and $-\Delta$, respectively. Observe that adding the two arcs does not introduce any cycles of negative-weight or zero-weight cycles with strict arcs, into the graph. Hence, the augmented system corresponding to $D'(\Delta)$ has a realization. Moreover, in every realization of $D'(\Delta)$, the distance between the left endpoints of x and y is Δ. By choosing different values of Δ, we contradict the rigidness of U. □

We can now generalize lemma 5.20:

Lemma 5.22. *Let Q be a realizable instance of MIO. For a non-negative p, let C be a union of* *cycles in $D(Q)$, each of weight less than p. If U is a strongly connected component in C then U is $|U|p$-rigid in Q.*

Proof. For vertices $x, y \in U$, by the definition of U, there is a simple path $P_0 : x = x_1, x_2, \ldots, x_k = y$ in C. Every edge $x_i x_{i+1}$ is in C, therefore there exists a path P_i in C from x_{i+1} to x_i s.t. $x_i x_{i+1}$ and P_i form a cycle in C. The concatenation of $P_{k-1}, P_{k-2}, \ldots, P_1$ is a path P from y to x in C. Moreover, the concatenation of P_0 and P is a cycle c in C (not necessarily simple) of weight at most $(k-1)p$. Let $d = w(P_0)$, $d' = w(P)$ and $q = |U|p$. Since $k \leq |U|$, we have established that $w(c) = w(P_0) + w(P) = d + d' \leq q$. Since Q is realizable, $d + d' \geq 0$. Hence, $-d \leq d' \leq q - d$. Summing the inequalities in $S(Q)$ along the two paths P_0 and P we get $l_y \leq l_x + d$ and $l_x \leq l_y + d'$, respectively. Any realization $\{l_x\}_{x \in V}$ of Q satisfies $S(Q)$, so $-d \leq l_x - l_y \leq d' \leq -d + q$. Hence, any two realizations are q-isometric. □

LEMMA 5.23. *Let $Q = (\prec, L, \phi)$ be a MIO instance, and let $Q' = (\prec, L')$ be the corresponding MIO* instance (obtained by the transformation in (18)). Suppose $U \subseteq V(\prec)$ is rigid in Q, and let $n = |V(\prec)|$. If $n^2\epsilon < \frac{1}{2}$ then U is $2n^2\epsilon$-rigid in Q'.*

Proof. The weight of each arc in $D(Q')$ changes by no more than 2ϵ, compared to $D(Q)$. Hence, the weight of every simple cycle changes by at most $2n\epsilon$. U is rigid in Q, hence, by lemma 5.21 it is contained in a strongly connected component W of a union of zero-weight cycles in $D(Q)$. W is also a union of simple zero-weight cycles in $D(Q)$. The weight of each such cycle in $D(Q')$ is at most $2n\epsilon$. Hence, by lemma 5.22, W is $2|W|n\epsilon$-rigid in Q', and so is U. □

We now return to the instance $\mathbb{P}$ generated by the reduction in the proof of theorem 5.16. Recall that $\mathbb{P}'$ is the instance obtained from $\mathbb{P}$ by the transformation (18). Suppose $\mathbb{P}'$ has a realization. Let $\prec$ be the corresponding interval order, and let $\mathbb{Q} = (\prec, L, \phi)$ and $\mathbb{Q}' = (\prec, L')$. Consider each of our gadgets: By lemma 5.4, every *Switch* is rigid in $\mathbb{Q}$. A slight modification of lemma 5.7 shows that every *Fetters* must be rigid in $\mathbb{Q}$ (since the directions of the *Switch*es are set). By lemma 5.8 every *Frame* is rigid in $\mathbb{Q}$, with the exception of its end β-intervals. Hence, each of these gadgets is $2n^2\epsilon$-rigid in $\mathbb{Q}'$, by lemma 5.23. This imposes, up to small additive shifts, the relative positions of the intervals in each vertex (or edge) sub-instance. Define the function *Col* as in (2). We shall show that the choice of ϵ makes these shifts sufficiently small so that the properties of the coloring are preserved.

LEMMA 5.24. *For each $i \in V$ there exist C_i, $|C_i| < 4n^2\epsilon$ s.t. $Col(i) + C_i \in \{0, 1, 2\}$.*

Proof. The proof is analogous to lemma 5.10: The relations (3)-(7) hold up to $\pm 2n^2\epsilon$. Hence, (8) holds up to $\pm 4n^2\epsilon$. □

LEMMA 5.25. *For every edge (i, j), $|Col(i) - Col(j)| \geq 1 - 8n^2\epsilon$.*

Proof. The proof is analogous to lemma 5.13: The relations (9)-(14) hold up to $\pm 2n^2\epsilon$. Relations (15)-(17) hold up to $\pm 8n^2\epsilon$, as they involve up to four differences of endpoint distances. □

Let $round(x)$ be the integer closest to x. Recall that $\epsilon < \frac{1}{20n^2}$. By lemma 5.24 $|Col(i) - round(Col(i))| < 0.2$. By lemma 5.25, for every edge (i, j): $|Col(i) - Col(j)| \geq 0.6$. Hence, $round(Col(i)) \neq round(Col(j))$. This proves that if there exist a realization to $\mathbb{P}'$, by rounding the colors to the nearest integer we obtain a proper 3-coloring. By lemma 5.15 this implies the existence of a realization to $\mathbb{P}$. Thus, $\mathbb{P}$ has a realization if and only if $\mathbb{P}'$ has one. Since the transformation described in (18) is polynomial, we conclude:

THEOREM 5.26. *MIG^* is NP-complete.* ■

5.4. Related Problems. In section 1 we have introduced the recognition problem of interval graph with individual lower and upper bounds on interval lengths (the BIG problem). Since MIG^* is a restriction of BIG and $DCIG$:

COROLLARY 5.27. *BIG and $DCIG$ are NP-complete.* ■

When restricted to interval graphs with depth 0 decomposition trees (see [23] for a definition of the decomposition tree), i.e., to prime interval graphs, the MIG problem can be solved in polynomial time, using the algorithm devised in section 3 for UCO graphs. This depth bound is indeed tight, namely, when allowing deeper decomposition trees the problem is NP-complete:

PROPOSITION 5.28.

MIG is NP-complete even when restricted to interval graphs with decomposition tree of depth 1.

Proof. We shall see that besides the *Switch*es $\delta_i(j)$ and $D_i(k)$, and the K_2 modules $\{c_3(k), c_1(k+1)\}$, $\{\gamma_3(i), \gamma_1(i+1)\}$, and $\{\gamma_3(n-1), c_1(0)\}$ there are no non-trivial modules in the interval graph constructed by the reduction in the proof of theorem 5.26: Let H be the graph obtained by contraction of the above modules. Suppose to the contrary that H contains a non-trivial module M, and suppose $v, u \in M$. If v, u are in the same vertex subgraph (or in the same edge subgraph) H_U, then $M \cap U$ is a non-trivial module in H_U, contradicting the primality of the vertex subgraph (and the edge subgraph). Hence, u, v are in different vertex/edge subgraphs. In this case, there are intervals in these subgraphs, which intersect only one out of u, v, in contradiction to M being a module. □

6. Acknowledgments. We thank Phil Green for illuminating conversations on the physical mapping problems which motivated this work. We also thank Garth Isaak for helpful discussions. We further thank the referees for their useful remarks.

REFERENCES

[1] R. K. Ahuja, T. A. Magnanti, and J. B. Orlin. *Network Flows: Theory, Algorithms and Applications.* Prentice Hall, Englewood Cliffs, NJ, 1993.

[2] J. F. Allen. Maintaining knowledge about temporal intervals. *Comm. ACM*, 26:832–843, 1983.

[3] J. F. Allen. *Reasoning about plans.* Morgan Kaufman, San Mateo, CA, 1991.

[4] P. Baldy and M. Morvan. A linear time and space algorithm to recognize interval orders. *Discrete Applied Mathematics*, 46:173–178, 1993.

[5] R. Bellman. On a routing problem. *Quarterly of Applied Mathematics*, 16:87–90, 1958.

[6] S. Benzer. On the topology of the genetic fine structure. *Proc. Nat. Acad. Sci. USA*, 45:1607–1620, 1959.

[7] K. S. Booth and G. S. Lueker. Testing for the consecutive ones property, interval graphs, and planarity using PQ-tree algorithms. *J. Comput. Sys. Sci.*, 13:335–379, 1976.

[8] A. V. Carrano. Establishing the order of human chromosome-specific DNA fragments. In A. D. Woodhead and B. J. Barnhart, editors, *Biotechnology and the Human Genome*, pages 37–50. Plenum Press, 1988.

[9] C. H. Coombs and J. E. K. Smith. On the detection of structures in attitudes and developmental processes. *Psych. Rev.*, 80:337–351, 1973.

[10] X. Deng, P. Hell, and J. Huang. Linear time representation algorithms for proper circular arc graphs and proper interval graphs. Technical report, School of Computing Science, Simon Fraser University, 1993.

[11] P. Fishburn. *Interval Orders and Interval Graphs.* Wiley, New York, 1985.

[12] P. Fishburn and R. L. Graham. Classes of interval graphs under expanding length restrictions. *J. Graph Theory*, 9:459–472, 1985.

[13] H. Gabow and R. E. Tarjan. Faster scaling algorithms for network problems. *SIAM J. Comput.*, 18:1013–1036, 1989.

[14] P. Gács and L. Lovász. Khachiyan's algorithm for linear programming. *Math. Prog. Study*, 14:61–68, 1981.

[15] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman and Co., San Francisco, 1979.

[16] P. C. Gilmore and A. J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canad. J. Math.*, 16:539–548, 1964.

[17] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Academic Press, New York, 1980.

[18] M. C. Golumbic, H. Kaplan, and R. Shamir. Graph sandwich problems. Technical Report 270-92, Computer Science Dept., Tel Aviv University, 1992. To appear in *Journal of Algorithms.*

[19] M. C. Golumbic and R. Shamir. Complexity and algorithms for reasoning about time: A graph-theoretic approach. *J. ACM*, 40:1108–1133, 1993.

[20] E. D. Green and P. Green. Sequence-tagged site (sts) content mapping of human chromosomes: Theoretical considerations and early experiences. *PCR Methods and Applications*, 1:77–90, 1991.

[21] E. D. Green and M. V. Olson. Chromosomal region of the Cystic Fibrosis gene in yeast artificial chromosomes: a model for human genome mapping. *Science*, 250:94–98, 1990.

[22] G. Hajös. Über eine art von graphen. *Intern. Math. Nachr.*, 11, 1957. problem 65.

[23] W-L. Hsu and T-H Ma. Substitution decomposition on chordal graphs and applications. In W-L. Hsu and R. C. T. Lee, editors, *Proc. 2nd Int. Symp on Algorithms (ISA '91)*, pages 52–60. Springer-Verlag, 1991. LNCS 557.

[24] G. Isaak. Discrete interval graphs with bounded representation. *Discrete Applied Mathematics*, 33:157–183, 1993.

[25] R. M. Karp. Mapping the genome: some combinatorial problems arising in molecular biology. In *Proc. 25th STOC*, pages 278–285. ACM Press, 1993.

[26] D. G. Kendall. Incidence matrices, interval graphs, and seriation in archaeology. *Pacific J. Math.*, 28:565–570, 1969.

[27] N. Korte and R. H. Möhring. Transitive orientation of graphs with side constraints. In H. Noltemeier, editor, *Proc. of the International workshop on Graphtheoretic concepts in Computer Science (WG '85)*, pages 143–160, Linz, 1985. Universitätsverlag Rudolf Trauner.

[28] N. Korte and R. H. Möhring. An incremental linear time algorithm for recognizing interval graphs. *SIAM J. Comput.*, 18:68–81, 1989.

[29] K. Nökel. *Temporally Distributed Symptoms in Technical Diagnosis.* Lecture Notes in Artificial Intelligence 517. Springer Verlag, 1991.

[30] J. B. Orlin and R. K. Ahuja. New scaling algorithms for the assignment and minimum cycle mean problems. *Mathematical Programming*, 54:41–56, 1992.

[31] C. Papadimitriou and M. Yannakakis. Scheduling interval ordered tasks. *SIAM J. Comput.*, 8:405–409, 1979.

[32] I. Pe'er and R. Shamir. Satisfiability problems on intervals and unit intervals. *Theoretical Computer Science*, 175:349–372, 1997. 2nd issue.

[33] F. S. Roberts. *Discrete Mathematical Models, with Applications to Social Biological and Environmental Problems.* Prentice-Hall, Englewood Cliffs, New Jersey, 1976.

[34] L. N. Shevrin and N. D. Filippov. Partially ordered sets and their comparability graphs. *Siberian Mathematics Journal*, 11:497–509, 1970.

[35] R. Stanley. Hyperplane arrangements, interval orders, and trees. *Proc. Nat. Acad. Sci.*, 93:2620–2625, 1996.

[36] S. A. Ward and R. H. Halstead. *Computation Structures.* MIT Press, 1990.

[37] A.B. Webber. Proof of an interval satisfiability conjecture. manuscript, Wesstern Illinois University, 1994.